

# Three-dimensional elastic body and a Naghdi type shell interaction modelling

---

Ljulj, Matko

Doctoral thesis / Disertacija

2020

*Degree Grantor / Ustanova koja je dodijelila akademski / stručni stupanj:* **University of Zagreb, Faculty of Science / Sveučilište u Zagrebu, Prirodoslovno-matematički fakultet**

*Permanent link / Trajna poveznica:* <https://urn.nsk.hr/urn:nbn:hr:217:661602>

*Rights / Prava:* [In copyright](#) / [Zaštićeno autorskim pravom.](#)

*Download date / Datum preuzimanja:* **2024-05-06**



*Repository / Repozitorij:*

[Repository of the Faculty of Science - University of Zagreb](#)





University of Zagreb

FACULTY OF SCIENCE  
DEPARTMENT OF MATHEMATICS

Matko Ljulj

**Three-dimensional elastic body and a  
Naghdi type shell interaction modelling**

DOCTORAL DISSERTATION

Zagreb, 2020.



University of Zagreb

FACULTY OF SCIENCE  
DEPARTMENT OF MATHEMATICS

Matko Ljulj

# **Three-dimensional elastic body and a Naghdi type shell interaction modelling**

DOCTORAL DISSERTATION

Supervisor:

prof. dr. sc. Josip Tambača

Zagreb, 2020.



Sveučilište u Zagrebu

PRIRODOSLOVNO–MATEMATIČKI FAKULTET  
MATEMATIČKI ODSJEK

Matko Ljulj

**Modeliranje međudjelovanja  
trodimenzionalnog elastičnog tijela i  
ljuske Naghdijeveog tipa**

DOKTORSKI RAD

Mentor:

prof. dr. sc. Josip Tambača

Zagreb, 2020.

# ACKNOWLEDGEMENTS

First of all, I would like to express my sincere gratitude to my mentor prof. dr. sc. Josip Tambača for more than just supervising my PhD thesis. His door has always been open for me and my questions, even silly or philosophical ones.

I am grateful to my family for everything they have done for me. They have always encouraged my interest in mathematics and it has never bothered them when I got obsessed by it.

I would also thank my friends who were always there for me, for reasons that are related to math and which are not.

And most importantly, to my wife Matea. I still cannot believe that our love can grow more and more every day. Thank you for everything.

# SUMMARY

In this thesis we analyse an interaction problem between two elastic three-dimensional bodies of which one is much thinner than the other. Furthermore, the thin body is assumed to be more stiff with elastic coefficients related to the small thickness parameter. We consider this problem in context of both linear and nonlinear elasticity. We analyse asymptotic behaviour of solutions of these problems and obtain different limit models in various regimes as thickness tends to zero. Furthermore, we propose both linear and nonlinear models given by three-dimensional elasticity equations and two-dimensional elastic plate equations as a boundary condition instead of the three-dimensional body equations that includes the thin body and investigate the asymptotic behaviour of the proposed model as the thickness parameter goes to zero. We prove that proposed model has the same asymptotic behaviour as firstly observed three-dimensional model that includes the thin body in the same regimes. By using proposed models, one can avoid numerical challenges that implementing a scheme for a three-dimensional model including a thin domain can cause as in similar situations.

The nonlinear model we propose is based on a nonlinear shell model we formulate and it is an additional contribution of the thesis. It is a nonlinear shell model of Naghdi type defined for shells which allow Lipschitz-continuous parametrizations of its middle surface, so it also models shells with middle surfaces having corners and folded plates and shells. Energy of the formulated model contains membrane, shear, drill and flexural terms, and (in appropriate regimes) it has the same asymptotic behaviour as models rigorously justified in the literature: membrane shell model, flexural shell model and constrained membrane plate model.

**Keywords:** interaction model, nonlinear elasticity, linearised elasticity, thin structure,

## Summary

---

justification,  $\Gamma$ -convergence, shells, Naghdi model, Cosserat model, membrane model, flexural model, Koiter model.

# SAŽETAK

U ovom radu analiziramo problem međudjelovanja dva elastična trodimenzionalna tijela od kojih je jedno mnogo tanje od drugoga. Nadalje, za tanko tijelo pretpostavljamo da je kruće, s koeficijentima elastičnosti ovisnima o parametru male vrijednosti koji predstavlja debljinu tankog tijela. Promatramo taj problem koristeći i linearnu i nelinearnu elastičnost. Analiziramo asimptotsko ponašanje rješenja tih problema i dobivamo različite granične modele u raznim režimima kada vrijednost tog parametra teži k nuli. Nadalje, predlažemo i linearan i nelinearan model opisan jednačbama trodimenzionalne elastičnosti i jednačbama elastične ploče kao rubnim uvjetom umjesto jednačbama trodimenzionalnog tijela koje uključuje tanko 3d tijelo, te istražujemo asimptotska svojstva predloženog modela kada parametar debljine teži k nuli. Dokazujemo da predloženi model ima jednako asimptotsko ponašanje kao trodimenzionalni model koji uključuje tanko 3d tijelo u istim režimima. Koristeći novopredložene modele moguće je izbjeći poteškoće kod numeričke implementacije trodimenzionalnih modela koji uključuju tanku domenu.

Predloženi nelinearni model baziran je na nelinearnom modelu elastične ljuske koji smo zadali i koji je dodatni doprinos ovog rada. To je nelinearan model ljuske Naghdijeva tipa definiran za ljuske kojima je središnja ploha parametrizirana Lipschitzovom funkcijom, pa također modelira i ljuske kojima središnja ploha sadrži kutove te presavinate ploče i ljuske. Energija u zadanom modelu sadrži membranske i fleksijske efekte te efekte smicanja i uvrtanja, te u odgovarajućem režimu ima ista asimptotska svojstva kao modeli koji su strogo izvedeni u literaturi: membranski model ljuske, fleksijski model ljuske i uvjetni membranski model ploče.

**Ključne riječi:** model interakcije, nelinearna elastičnost, linearizirana elastičnost,



## Sažetak

---

tanka struktura, rigorozno opravdanje,  $\Gamma$ -konvergencija, model ljuske, Naghdijev model, Cosseratov model, membranski model, fleksijski model, Koiterov model.

# CONTENTS

<b>Introduction</b>	<b>1</b>
1 Notation . . . . .	8
 <b>I Linear models of elastic bodies interaction</b>	 <b>11</b>
 <b>1 A Naghdi type linear shell model</b>	 <b>12</b>
1.1 Introduction . . . . .	12
1.2 Geometry . . . . .	14
1.3 The linear shell model . . . . .	16
 <b>2 Linear 3d–thin 3d and 3d–2d model</b>	 <b>19</b>
2.1 Introduction . . . . .	19
2.2 Description of the problem . . . . .	20
2.3 Limits of the 3d–thin 3d model . . . . .	25
2.3.1 3d–thin 3d model: case $-1 \leq \alpha < 0$ . . . . .	30
2.3.2 3d–thin 3d model: case $\alpha = 0$ . . . . .	31
2.3.3 3d–thin 3d model: case $0 < \alpha < 2$ . . . . .	33
2.3.4 3d–thin 3d model: case $\alpha = 2$ . . . . .	34
2.3.5 3d–thin 3d model: case $\alpha > 2$ . . . . .	40
2.4 Limits of the 3d–2d model . . . . .	41
2.4.1 3d–2d model: case $-1 \leq \alpha < 0$ . . . . .	43
2.4.2 3d–2d model: case $\alpha = 0$ . . . . .	45
2.4.3 3d–2d model: case $0 < \alpha < 2$ . . . . .	47
2.4.4 3d–2d model: case $\alpha = 2$ . . . . .	48

2.4.5	3d–2d model: case $\alpha > 2$ . . . . .	50
<b>II</b>	<b>Nonlinear models of elastic bodies interaction</b>	<b>52</b>
<b>3</b>	<b>Asymptotically obtained nonlinear plate models</b>	<b>53</b>
3.1	Problem setup . . . . .	53
3.2	The main result . . . . .	57
3.3	Technical results and proofs . . . . .	61
<b>4</b>	<b>Nonlinear 3d–thin 3d model</b>	<b>67</b>
4.1	Problem setup . . . . .	67
4.2	The main result . . . . .	70
4.3	Technical results . . . . .	73
4.4	Proof of Theorem 4.2.1 . . . . .	82
<b>5</b>	<b>A Naghdi type nonlinear shell model</b>	<b>86</b>
5.1	Introduction . . . . .	86
5.2	Definition of the model . . . . .	90
5.3	Properties of the model . . . . .	94
5.3.1	On sets of admissible functions . . . . .	94
5.3.2	The flexural shell model . . . . .	95
5.3.3	The Koiter shell model . . . . .	98
5.3.4	Linearization of the model . . . . .	100
5.3.5	Differential formulation . . . . .	102
5.4	Convergence in regimes . . . . .	104
5.4.1	Technical lemmas . . . . .	109
5.4.2	Membrane shell model - convergence for the case $\alpha = 0$ . . . . .	112
5.4.3	Membrane shell model – further properties . . . . .	116
5.4.4	Flexural shell model - convergence for the case $\alpha = 2$ . . . . .	123
5.4.5	Constrained membrane plate model - convergence for the case $\alpha \in \langle 0, 5/3 \rangle$ . . . . .	132
5.4.6	Convergence for cases $\alpha \in \langle -1, 0 \rangle$ and $\alpha \in \langle 2, +\infty \rangle$ . . . . .	140

5.5	Technical lemmas for matrices . . . . .	145
<b>6</b>	<b>Nonlinear 3d–2d model</b>	<b>151</b>
6.1	Problem setup . . . . .	151
6.2	The main result . . . . .	153
6.3	Proof of Theorems 6.2.1 and 6.2.2 . . . . .	158
6.4	Comparison with the 3d–thin 3d model . . . . .	162
	<b>Conclusion</b>	<b>165</b>
	<b>Appendix</b>	<b>167</b>
A	The main technical lemma . . . . .	167
	<b>Curriculum Vitae</b>	<b>191</b>

# INTRODUCTION

Interaction of two continua appears in a vast number of real life situations and is of big interest in many applications. The same holds for interaction of structures made of two continua in which one of them is much thinner than the other. Examples are countless: flow of a fluid through a pipe, bridges, roads and various other elements in construction, interaction of Earth's layers with application in geophysics, etc. In this thesis we focus on such three-dimensional structures in which both layers are modelled as elastic bodies.

In numerical solving in purposes of research and applications of mathematical models, the thickness of a medium plays an important role. In thin domains mesh for solving mathematical models has to be very fine in order to obtain regular mesh. Very similar problem arises in the interaction of two continua where one of them is much thinner than the other. In this situation the thin layer has to be modelled on a fine mesh, which also causes problems in numerical coupling to the other continuum.

To better describe this numerical challenge, let us observe the following example. A two-dimensional body of length 1 and height  $1 + h$ ,  $h \ll 1$  is modelled by linearised elasticity equations. It is clamped at the bottom, and a force is applied at the top. This body is made of two different materials, on the lower part of dimension  $1 \times 1$  and on the upper part of dimension  $1 \times h$ . Difference between those materials is seen through different Lamé coefficients  $\lambda, \mu$ . Ratio of coefficients for the thin part and for the lower part is assumed to be  $h^{-1-\alpha}$ , where  $\alpha$  is a real parameter. Such problem is observed in [63].

In case  $\alpha = -1$ , elastic properties of the thin part and the thick part coincide. Due to continuous dependance on the geometry, it is expected that for small value of thickness  $h$  the thin part does not contribute in any way to the structure. However, for values  $\alpha > -1$  we expect some contribution of the thin layer. We numerically solve the described

problem for various values of that parameter. We present behaviour of the structure for values  $\alpha = -1, 0, 1, 2, 3$  and  $h = 2^{-4}$ , for forcing term equal to

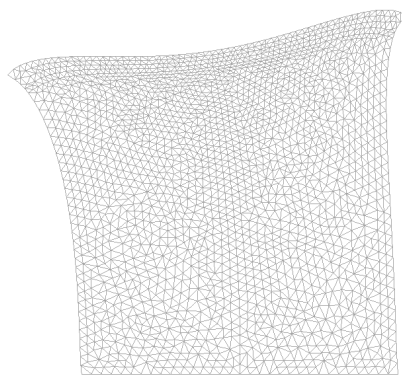
$$\mathbf{f}(x_1) = 10^6 \left( \left( x_1 - \frac{1}{2} \right) \mathbf{e}_1 + \left( 4(x_1 - 0.2)(x_1 - 0.7) - \frac{7}{75} \right) \mathbf{e}_2 \right). \quad (1)$$

The results are in Figure 1. In the Figure 1.(a) example we see the confirmation of our mathematical based conclusions – the thin part of the body is quite deformed, but in essence the top of the lower part has the same shape as the thin part. In the limit there will be no contribution of this material to the total structure.

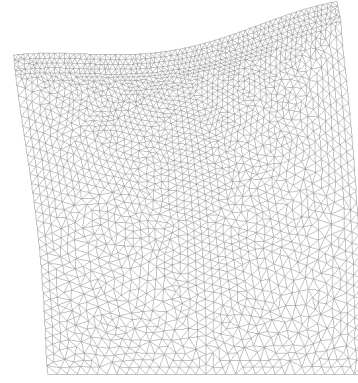
As  $\alpha$  gets larger, we observe different behaviour of the upper part of domain. What all cases have in common is that the larger  $\alpha$  implies higher level of rigidity of the thin part. Two effects can be seen with a bare eye. Firstly, for values  $\alpha = -1, 0$  the thin part of the domain is getting more resistant to stretching, until in cases  $\alpha = 1, 2, 3$  the thin part gets approximately of the same length as in undeformed geometry (it is easily seen by looking at the top side length of the deformed rectangle). Secondly, in cases  $\alpha = 0, 1, 2$  the same thin part is becoming reluctant to bend, until the case  $\alpha = 3$  is reached. In this case it is neither bent nor stretched, it is again of the form of the (thin) undeformed rectangle. In elasticity theory, those two effects (stretching and bending) are also called membrane and flexural effects and are related to the plate models and energy terms of the same name.

Looking at the same figures, let us also observe the mesh density. All problems are solved on the same mesh. At the top the mesh is dense to obtain regular triangulation and precise enough approximation, which is common for problems defined on thin domains. However, due to coupling, this dense mesh is spread partially to a lower part of the domain as well. Thus that part of domain is meshed denser than it can be when the problem is defined only on this lower part of domain (without the thin layer). This makes numerical algorithm more challenging and time-consuming than we would like.

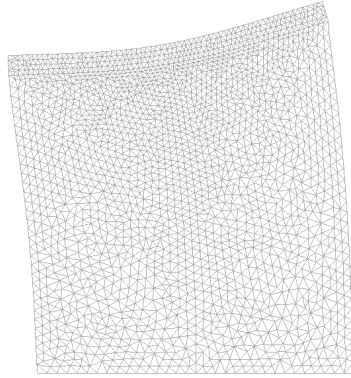
One solution to approach this drawback (apart from treating it with some numerical methods designed for non-matching meshes) is, since  $h \ll 1$ , to rigorously derive the limit model when  $h \rightarrow 0$ . More precisely, one can approximate the solution of this problem by a limit of solutions of problems when  $h \rightarrow 0$ , which is defined as a solution to a PDE defined on a  $1 \times 1$  square, i.e. without the thin layer. In this way numerical drawbacks are avoided.



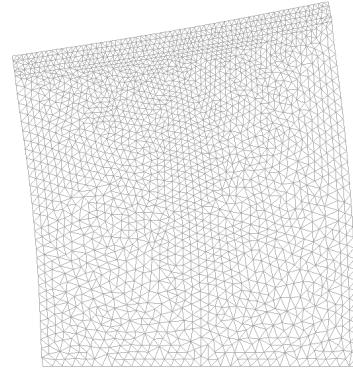
(a)  $\alpha = -1$



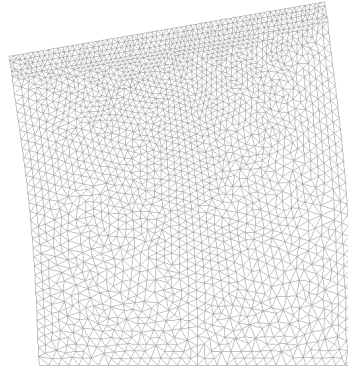
(b)  $\alpha = 0$



(c)  $\alpha = 1$



(d)  $\alpha = 2$



(e)  $\alpha = 3$

Figure 1: Numerical approximation of solutions for deformation of two-dimensional multimaterial body, for various values of parameters  $\alpha$ , thickness  $h = 2^{-4}$  and the forcing only at the top given in (1).

However, for this we have to rigorously derive mentioned limit model when  $h \rightarrow 0$ . We show that for different values of  $\alpha$  different limit models are obtained, considering either the membrane or flexural energy model on the thin part of the domain. On the other hand, by observing numerical solutions in Figure 1 we see that both membrane and flexural effects can happen. Also, in real life applications, we do not deal with a regime (related to  $\alpha$ ), but only one fixed set of parameters for this problem, so it is sometimes not clear a priori which regime we relate to and consequently which limit model we should choose.

For the same reasons, in the same paper [63] a 2d–1d model is proposed, dependant on the parameter  $h \ll 1$ , representing the thickness of the thin layer at the top. It takes into account the 2d elastic energy of the larger body and 1d model including membrane and flexural energies for the layer at the top. Its main properties are that it is easy to numerically approximate it and that in all regimes (related to the parameter  $\alpha$ ) when  $h \rightarrow 0$  the particular limit model is the same the limit model obtained in the rigorous derivation of the 2d problem related to the same  $\alpha$ . In this way, neither do we have to numerically approximate the problem on a dense mesh, nor do we have to choose  $\alpha$  (and the limit model for that  $\alpha$ ) in advance – both were major drawbacks of other ways of approaching to our problem.

This motivates the problem setting for this thesis.

In this thesis we observe a structure made of two elastic bodies. One is situated in the closure of  $\Omega^0 := \langle 0, 1 \rangle \times \langle 0, 1 \rangle \times \langle -1, 0 \rangle$  and another (thin) body is situated in the closure of  $\Omega^h := \langle 0, 1 \rangle \times \langle 0, 1 \rangle \times \langle 0, h \rangle$ , for a small parameter  $h$ . The thinner body is assumed to be more stiff than the larger body, again by an order of magnitude  $h^{-\alpha-1}$ , as in the numerical example. A force is applied to the top of the structure (at  $\langle 0, 1 \rangle \times \langle 0, 1 \rangle \times \{h\}$ ), and the whole structure is clamped at the bottom (at  $\langle 0, 1 \rangle \times \langle 0, 1 \rangle \times \{-1\}$ ).

Firstly, for each structure we define a model to which the solution is the deformation of the structure, for each parameter  $\alpha$  and  $h$ . Secondly, for various parameters  $\alpha$  we will rigorously derive a limit model when  $h \rightarrow 0$ . Finally, as in the numerical example, we will propose a 3d–2d model and prove that it has the same asymptotic properties as original model.

We will perform announced analysis in terms of linearised elasticity (Part I) and in



terms of nonlinear elasticity (Part II). Key differences from the numerical example are that in the thesis the emphasis is on the theoretical approach (thus we will not implement numerical algorithms), and that the problem is observed on 3d domains instead of the 2d domains as in the numerical example. On the other hand, even though Figure 1 shows numerically obtained (and not mathematically justified) approximation of solutions and even though it describes the situation in two dimensions, the analogous behaviour is expected in our three-dimensional modelling in this thesis.

The 3d–2d model we will propose takes into account different effects that a thin elastic body can have, for example membrane and flexural effects mentioned earlier. For this reason, in Part I (in case of linearised elasticity) we will base our 3d–2d model on an existing linear shell model of the Naghdi type ([87]). In the case of nonlinear modelling (Part II) we need a nonlinear generalization of this shell model, which does not exist in the literature so far. Thus in this thesis in Chapter 5 we propose a nonlinear shell model of the Naghdi type, having similar properties as the linear model from [87] and that could be considered as a nonlinear generalisation of that model. Apart from the rigorous asymptotic derivation and obtaining 3d–2d limit models for various values of parameter  $\alpha$  and apart from proposing adequate 3d–2d model having the same asymptotic properties as the 3d–thin 3d model, both in linear and nonlinear case, proposition of the new nonlinear shell model is another major contribution of this thesis.

Main contributions of this thesis are:

- We rigorously derive limit models of 3d–thin 3d problem when the thickness parameter  $h$  tends to zero, in as many regimes (regarding  $\alpha$ ) as possible, in both linear and nonlinear case. In the existing literature within the scope of linearised elasticity the asymptotics is discussed in a bit different setting, so the arguments are quite different. In the nonlinear setting only particular values of  $\alpha$  are discussed in the literature.
- We formulate a new nonlinear shell model of the Naghdi type and investigate and prove some of its properties. The most important are, as a difference to majority of shell models in the literature, that the model is well defined for shells with little regularity ( $W^{1,\infty}(\omega; \mathbb{R}^3)$  parametrisations of the middle surface) and that it asymp-

totically (with respect to the thickness of the shell), in corresponding regimes, behaves as 3d equations (this is analysed in rigorous derivation of shell models in the literature).

- We propose 3d–2d model and prove that its limit models when thickness parameter  $h$  tends to zero coincide with limit models for 3d–thin 3d problem, in various regimes (regarding  $\alpha$ ), in both linear and nonlinear case. To our knowledge, this model is novel and does not exist in the literature.

More detailed list of contributions can be found in Conclusion. Parts of this thesis already resulted in published or submitted articles. The whole linear modelling part is published in [62], and the proposition of the nonlinear shell model with the majority of its properties is presented in [61].

## Literature overview

Modelling of thin structures in context of elasticity equations is important area within continuum mechanics not only because of more simple structure of the plate and shell equations but also because they are more simple to analyse and it is easier to build numerical schemes for them. For a modern general theory of nonlinear elastic plate and shell theory see [6, 20, 21, 28]. The nonlinear plate and shell theories can be classified in two categories. A first category is made of two-dimensional models that are obtained from the three-dimensional equations of shell-like bodies by letting the thickness of the body tend to zero. One of the first attempts of such approach was in [37] by means of asymptotic formalism. Later on, various plate models (and afterwards shell models) were justified by means of  $\Gamma$ –convergence. Now there exists a hierarchy of models depending on the order of the elastic energy, and works in which those limit models are investigated in more details, see [30, 31, 38–40, 51–54, 57–60, 74]. The second category of plate and shell models are obtained from the three-dimensional models by taking a priori assumptions on the range of admissible deformations and stresses that a thin body can have, such as Cosserat assumptions, Koiter assumptions etc. For foundation of the Naghdi and Koiter models of shells see [47, 67]. Precise justifications and generalizations of the Koiter model can be found in [13, 22, 26, 27, 29, 65, 66, 84], and some existence results for the Koiter and some

other nonlinear shell models can be found in [3, 4, 25]. Some shell models coming from nonlinear theory based on Cosserat assumptions also can be found in [11, 69–72].

The problems of interaction of different continua or continua of different dimensions are the area of great interest as well. Presently in the literature dominant are problems that include linear modelling. The asymptotics of the linear 3d–thin 3d problem in a very similar setting as ours in the Section 2.3 was considered in [18], where the author obtained the convergence results in all cases for the parameter  $\alpha$ , however the setting used in this thesis is slightly different which results in different proofs. Essentially the same linear 3d–thin 3d problem asymptotics, but only for  $\alpha = 2$ , is considered in [23] and elaborated in [20]. In the case of shells the 3d–2d asymptotics is discussed in [9]. Due to the curved geometry the considered problem is more complicated but gives the asymptotics of the 3d–2d problem only for  $\alpha = 0$  and  $\alpha = 2$ . One can also see [1] for the variational approach to the thin inclusion problem in linear elasticity and [8] in both linear and nonlinear elasticity. For hyperelastic materials and  $\alpha = 0$  the asymptotics of the 3d–thin 3d problem is discussed in [35] by  $\Gamma$ –convergence techniques. A similar analysis for micropolar elastic media is done in [82]. Interaction of the viscous fluid and the linearised elasticity/elastic plate for the same choice of the stiffness of the elastic coefficients ( $\alpha = 2$ ) and thin structure is considered in [64, 73]. For inclusion of 1d elastic and rigid bodies in 2d linearised elasticity see [46]. An example from electromagnetism can be found in [77]. Linear problems of interaction of thin 3d bodies and elastic plates and shells in cases  $\alpha = 0$  and  $\alpha = 2$  are observed in [10]. Contact problems between linear 3d body and rigid plates are observed in [36]. This topic is also related to the problem of modelling of joints within both nonlinear and linearised elasticity, see [43, 49] for thin elastic interfaces, linear and nonlinear, isotropic and functionally graded, with or without constraints, see [33, 41, 55, 56, 75]. Interaction of linear elastic plates is observed in [42] and [83]. For piezoelectric interfaces see [79, 81] and for magneto-electro-thermo-elastic see [80]. For application of elastic interfaces to geophysics, see [78].

## Chapter overview

The thesis is divided in two parts. In Part I the problem explained earlier is observed

from the perspective of linearised elasticity and Part II is dedicated to nonlinear elasticity. In Chapter 1 we present a linear Naghdi type model (from [87]) on which we will base the linear 3d–2d model. In the same chapter some definitions regarding the geometry of the shell are also announced. In Chapter 2 we present the first contribution of the thesis. There we rigorously derive limit models in all cases for  $\alpha \geq -1$  of interaction of 3d body and thin 3d body (3d–thin 3d) in aspect of linear elasticity. Also, we propose a linear 3d–2d model and prove its asymptotic properties.

In Part II we observe the problem using nonlinear elasticity modelling. We start with presenting known results in asymptotically derived nonlinear plate models in Chapter 3. In Chapter 4 we rigorously derive limit models of the 3d–thin 3d problem in various cases depending on the real parameter  $\alpha$ . Chapter 5 is dedicated to the new nonlinear shell model of Naghdi type. We propose the model and prove its properties. Finally, in Chapter 6 we propose a nonlinear 3d–2d model based on the new shell model from Chapter 5, and compare its asymptotics to the one in Chapter 4. At the end of the thesis, in Appendix A, there is a technical density lemma used in Chapter 5 and Chapter 6.

## 1. NOTATION

At the beginning, we introduce the notation we are going to use throughout the thesis. Sets of natural numbers (positive integers) and real numbers are denoted by  $\mathbb{N}$  and  $\mathbb{R}$ , respectively. We also use notation  $\overline{\mathbb{R}} := [-\infty, +\infty]$ . If not stressed differently, Latin letters  $i, j, k$  denote indices from the set  $\{1, 2, 3\}$ , and for indices from the set  $\{1, 2\}$  we use Greek letters  $\beta, \beta_1, \beta_2$ . The Greek letter  $\alpha$  is not used here since it is reserved for a parameter related to the ratio of elasticity properties between the 3d and thin 3d part of the structure explained earlier.

Small bold letter  $\mathbf{x} = (x_1, x_2, x_3)$ , if not noted differently, is reserved for a point  $(x_1, x_2, x_3)$  in the space  $\mathbb{R}^3$ , while  $\mathbf{x}' = (x_1, x_2)$  is reserved for a point  $(x_1, x_2)$  in  $\mathbb{R}^2$ . We will often use  $\mathbf{x}'$  as abbreviation for first two coordinates of the point  $\mathbf{x}$ .

Most other small bold latin and greek letters are used for vectors or vector functions. Capital bold latin letters are reserved for matrices.

The set of  $m \times n$  real matrices is denoted by  $\mathbb{M}^{m \times n}$ . The determinant and trace of the

matrix  $\mathbf{X}$  are denoted by  $\det(\mathbf{X})$  and  $\text{tr}(\mathbf{X})$ , respectively. The identity matrix is denoted by  $\mathbf{I}$ .

Very often we will use a particular subset of  $\mathbb{M}^{3 \times 3}$ :

$$\text{SO}(3) := \{\mathbf{S} \in \mathbb{M}^{3 \times 3} : \mathbf{S}^T \mathbf{S} = \mathbf{S} \mathbf{S}^T = \mathbf{I}, \det \mathbf{S} = 1\};$$

such matrices are referred as rotations. It is clearly a compact subset of  $\mathbb{M}^{3 \times 3}$ .

For vector and matrix norms, the following notation is used

- $\|\mathbf{a}\|$  denotes the 2–norm of a vector  $\mathbf{a} \in \mathbb{R}^n$ ;
- $\|\mathbf{A}\|_F$  denotes the Frobenius norm of a matrix  $\mathbf{A} \in \mathbb{M}^{m \times n}$ , defined by

$$\|\mathbf{A}\|_F = (\text{tr}(\mathbf{A}^T \mathbf{A}))^{1/2};$$

- $\|\mathbf{A}\|_2$  denotes the induced operator 2–norm of a matrix  $\mathbf{A} \in \mathbb{M}^{m \times n}$ , defined by

$$\|\mathbf{A}\|_2 = \sup_{\mathbf{y} \in \mathbb{R}^n, \mathbf{y} \neq 0} \frac{\|\mathbf{A}\mathbf{y}\|}{\|\mathbf{y}\|}.$$

Scalar (dot, inner) product and vector (cross) product of vectors  $\mathbf{a}$  and  $\mathbf{b}$  are denoted by  $\mathbf{a} \cdot \mathbf{b}$  and  $\mathbf{a} \times \mathbf{b}$ , respectively. The Frobenius scalar product of matrices  $\text{tr}(\mathbf{A}^T \mathbf{B})$  is denoted by  $\mathbf{A} \cdot \mathbf{B}$ .

For matrices  $\mathbf{X}, \mathbf{Y} \in \mathbb{M}^{m \times n}$  of the same type we define  $\text{dist}(\mathbf{X}, \mathbf{Y}) = \|\mathbf{X} - \mathbf{Y}\|_F$ . For a set of matrices  $\mathcal{S} \subset \mathbb{M}^{m \times n}$  and a matrix  $\mathbf{X} \in \mathbb{M}^{m \times n}$  we define

$$\text{dist}(\mathbf{X}, \mathcal{S}) := \inf_{\mathbf{Y} \in \mathcal{S}} \text{dist}(\mathbf{X}, \mathbf{Y}).$$

Most often we will use  $\text{dist}$  function for  $\mathcal{S} := \text{SO}(3)$ . Since  $\text{SO}(3)$  is a compact set and  $\text{dist}$  is a continuous function, in definition of  $\text{dist}(\mathbf{X}, \text{SO}(3))$  the "inf" can be substituted by "min". Thus, for each  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$  there exists the "closest" rotation  $\mathbf{R}_{\mathbf{X}} \in \text{SO}(3)$  (at least one) for which  $\text{dist}(\mathbf{X}, \text{SO}(3)) = \text{dist}(\mathbf{X}, \mathbf{R}_{\mathbf{X}})$ . If  $\mathbf{X}$  has a positive determinant and  $\mathbf{X} = \mathbf{R}\mathbf{U}$  is its polar decomposition, by simple calculations we have  $\text{dist}(\mathbf{X}, \text{SO}(3)) = \|\mathbf{U} - \mathbf{I}\|_F$ . If a condition on determinant is dropped, we still have a bound  $\text{dist}(\mathbf{X}, \text{SO}(3)) \geq \|(\mathbf{X}^T \mathbf{X})^{1/2} - \mathbf{I}\|_F$ .

For a matrix  $\mathbf{A}$  the numbers  $\lambda_i(\mathbf{A})$  and  $\sigma_i(\mathbf{A})$  denote  $i^{\text{th}}$  largest eigenvalue and singular value, respectively. Sometimes, for simplicity, self-explanatory notations  $\lambda_{\min}(\mathbf{A})$ ,  $\lambda_{\max}(\mathbf{A})$ ,  $\sigma_{\min}(\mathbf{A})$  and  $\sigma_{\max}(\mathbf{A})$  are also used.

We will use Lebesgue function spaces  $L^p(\Omega; \mathbb{R}^n)$  on different domains and codomain dimensions. For a power ( $p$  in the above example) we use real constants (most often 2 and 4) or letters  $p$  and  $q$ . We denote their Hölder conjugates by  $p' = (1 - 1/p)^{-1}$  and  $q' = (1 - 1/q)^{-1}$ . In the case that the codomain is a subset of matrices  $\mathbb{M}^{m \times n}$ , the Frobenius norm is used in the definition of the  $L^p(\Omega; \mathbb{R}^{m \times n})$  norm. (Weak) partial derivatives of a function  $\boldsymbol{\eta}$  are denoted by  $\partial_1 \boldsymbol{\eta}, \partial_2 \boldsymbol{\eta}, \dots$ , and the differential of order  $n$  is denoted by  $D^n \boldsymbol{\eta}$ . The gradient is denoted by  $\nabla \boldsymbol{\eta}$ . In case when the gradient is taken with respect of just two first variables or the function  $\boldsymbol{\eta}$  is defined on a subset of  $\mathbb{R}^2$ , we will emphasise it by notation  $\nabla' \boldsymbol{\eta} := \begin{bmatrix} \partial_1 \boldsymbol{\eta} & \partial_2 \boldsymbol{\eta} \end{bmatrix}$ . We will also use Sobolev functions spaces: if  $\Omega \subset \mathbb{R}^m$  we define

$$W^{k,p}(\Omega; \mathbb{R}^n) = \{\boldsymbol{\eta} \in L^p(\Omega; \mathbb{R}^n) : \partial_i \boldsymbol{\eta} \in W^{k-1,p}(\Omega; \mathbb{R}^n), i = 1, \dots, m\}$$

(with the convention  $W^{0,p}(\Omega; \mathbb{R}^n) \equiv L^p(\Omega; \mathbb{R}^n)$ ). Also,  $H^k(\Omega; \mathbb{R}^n) := W^{k,2}(\Omega; \mathbb{R}^n)$ .

In Part II of the thesis we will often use  $\Gamma$ -convergence. For a metric space  $X$ , a family of functionals  $(F_n)_{n \geq 1}$ ,  $F_n : X \rightarrow \overline{\mathbb{R}}$   $\Gamma$ -converges to a functional  $F : X \rightarrow \overline{\mathbb{R}}$  in the topology of  $X$  if the following claims hold:

**The "lim inf" part:** For all  $x \in X$  and for any  $(x_n)_{n \geq 1} \subset X$ ,  $x_n \rightarrow x$  in  $X$  holds

$$\liminf_{n \rightarrow \infty} F_n(x_n) \geq F(x);$$

**The "lim sup" part:** For all  $x \in X$  there exists  $(x_n)_{n \geq 1} \subset X$ ,  $x_n \rightarrow x$  in  $X$  such that

$$\limsup_{n \rightarrow \infty} F_n(x_n) \leq F(x).$$

In that case, we will denote  $F = \Gamma((F_n)_{n \geq 1})$ . If the sequence of functionals is a constant sequence (i.e.  $F_n = F_0$  for all  $n \in \mathbb{N}$ ), then we will denote  $F = \Gamma - F_0$ . In case that  $X$  is a Lebesgue function space, it is known that  $F$  is then the weakly lower semicontinuous envelope of  $F_0$ .

# **Part I**

## **Linear models of elastic bodies interaction**

# 1. A NAGHDI TYPE LINEAR SHELL MODEL

## 1.1. INTRODUCTION

In the first part of the thesis we deal with interaction between a 3d elastic body and a thin 3d body. Apart from determining limit models (with respect to  $h \rightarrow 0$ , where  $h$  is the thickness of the thin 3d part), we would like to propose an adequate 3d–2d model that has the same asymptotic properties as the 3d–thin 3d model. For that reason in this chapter we present a linear shell model on which our 3d–2d model is based, and in Chapter 2 we perform all asymptotic analysis.

This shell model we use is from [87]. It accounts for all three basic notions of the deformation of shells, namely stretching, shear and bending, like in the classical Naghdi shell model, see [67]. However, the model we use has simple structure, and one can build a numerical algorithm for it simpler than in the case of the classical Naghdi model, see [87] for details. It is a six parameter model. In the case of planar case of undeformed geometry of the middle surface, for the flexural deflection we obtain exactly the Reissner–Mindlin plate model, but the model also contains the in-plane deformation and the in-plane drill. These properties suggest that it will suit us for 3d–2d model we will propose in Chapter 2.

The main features of the model are:

- The model is formulated for the unknown  $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}})$  in a subset  $V_N(\omega)$  of  $H^1(\omega; \mathbb{R}^3) \times H^1(\omega; \mathbb{R}^3)$  defined by boundary conditions ( $\omega \subset \mathbb{R}^2$  being open, bounded with Lipschitz boundary).



- The proof of existence and uniqueness of the solution has a very simple proof without use of delicate Korn's type estimate.
- The model is well defined for the middle surface parametrised by  $\boldsymbol{\theta} \in W^{1,\infty}(\omega; \mathbb{R}^3)$  (and thus the model for shells with middle surfaces with corners (or folded plates or shells) is inherently built into the model).
- The energy of the model contains the membrane, transverse shear and flexural terms which are of different order with respect to the thickness  $h$  of the shell.
- For smooth geometry the solution of the model in the elliptic membrane and flexural regime tends to the solution of the corresponding shell model, when thickness  $h$  tends to zero.
- The model can be seen as a small perturbation of the classical Naghdi shell model.
- The solution of the model continuously depends on the change in the geometry (with respect to parametrization  $\boldsymbol{\theta}$  in  $W^{1,\infty}(\omega; \mathbb{R}^3)$ ).
- The model can be seen as the special Cosserat shell model with a single director for a particular linear constitutive law.

## 1.2. GEOMETRY

In order to formulate shell we firstly give necessary geometry definitions for the middle surface. Let  $\omega \subset \mathbb{R}^2$  be an open bounded and simply connected set with Lipschitz-continuous boundary  $\gamma$  and let an injective function  $\boldsymbol{\theta} \in C^3(\bar{\omega}; \mathbb{R}^3)$  be the parametrization of the middle surface of the shell, see Figure 1.1. Points in  $\bar{\omega}$  we denote by  $\mathbf{x}' = (x_\beta)$  and let  $\partial_\beta := \partial/\partial x_\beta$ . The assumption that  $\mathbf{a}_\beta(\mathbf{x}') = \partial_\beta \boldsymbol{\theta}(\mathbf{x}')$  are linearly independent at all points  $\mathbf{x}' \in \bar{\omega}$  assures that  $\boldsymbol{\theta}$  parametrizes two-dimensional surface  $S = \boldsymbol{\theta}(\bar{\omega})$  in  $\mathbb{R}^3$ .

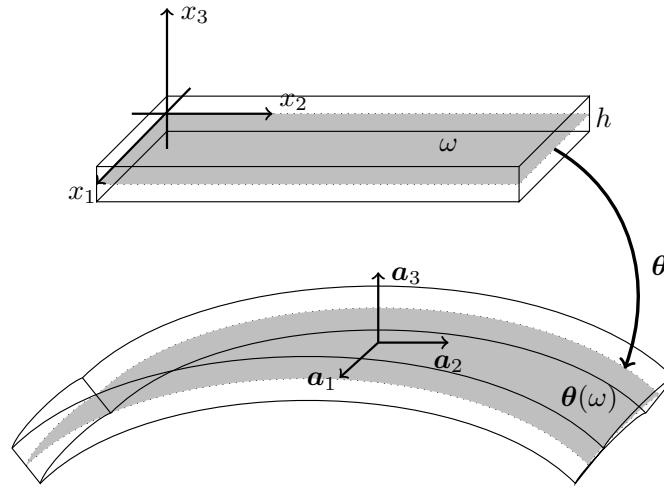


Figure 1.1: Parametrization of the undeformed shell.

Vectors  $\mathbf{a}_1(\mathbf{x}')$ ,  $\mathbf{a}_2(\mathbf{x}')$  form the covariant basis of the tangent plane of  $S$  at  $\boldsymbol{\theta}(\mathbf{x}')$ , and the contravariant basis of the same plane is given by  $\mathbf{a}^\beta(\mathbf{x}')$ ,

$$\mathbf{a}^{\beta_1}(\mathbf{x}') \cdot \mathbf{a}_{\beta_2}(\mathbf{x}') = \delta_{\beta_1, \beta_2}.$$

We also define a vector

$$\mathbf{a}_3(\mathbf{x}') = \mathbf{a}^3(\mathbf{x}') = \frac{\mathbf{a}_1(\mathbf{x}') \times \mathbf{a}_2(\mathbf{x}')}{\|\mathbf{a}_1(\mathbf{x}') \times \mathbf{a}_2(\mathbf{x}')\|}.$$

Note that sets  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$  and  $\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\}$  form bases for  $\mathbb{R}^3$ . The first fundamental form, or the metric tensor in covariant  $\mathbf{A}_c = (a_{\beta_1 \beta_2})$  or contravariant  $\mathbf{A}^c = (a^{\beta_1 \beta_2})$  components of the surface  $S$  is given respectively by  $a_{\beta_1 \beta_2} = \mathbf{a}_{\beta_1} \cdot \mathbf{a}_{\beta_2}$ ,  $a^{\beta_1 \beta_2} = \mathbf{a}^{\beta_1} \cdot \mathbf{a}^{\beta_2}$ .

Note that we have  $\mathbf{A}^c = \mathbf{A}_c^{-1}$  and

$$\mathbf{a}_1 \times \mathbf{a}_3 = -\sqrt{a} \mathbf{a}^2, \quad \mathbf{a}_2 \times \mathbf{a}_3 = \sqrt{a} \mathbf{a}^1. \quad (1.2.1)$$

The area element along  $S$  is then  $\sqrt{a}d\mathbf{x}'$ , where  $a := \det \mathbf{A}_c$ . It is easy to prove that  $\mathbf{A}^c$ ,  $\mathbf{A}_c$  and  $a$  are uniformly positive definite, i.e., that

$$\operatorname{ess\,inf}_{\mathbf{x}' \in \omega} \lambda_{\min}(\mathbf{A}^c(\mathbf{x}')), \operatorname{ess\,inf}_{\mathbf{x}' \in \omega} \lambda_{\min}(\mathbf{A}_c(\mathbf{x}')), \operatorname{ess\,inf}_{\mathbf{x}' \in \omega} a(\mathbf{x}') > 0. \quad (1.2.2)$$

However, for less regular parametrizations, i.e.  $\boldsymbol{\theta} \in W^{1,\infty}(\omega; \mathbb{R}^3)$ , this will be an assumption. For more details see [21, 85–87].

### 1.3. THE LINEAR SHELL MODEL

In this section we formulate the shell model of Naghdi type from [87]. The model is defined for middle surfaces parametrised by a  $\boldsymbol{\theta} \in W^{1,\infty}(\omega; \mathbb{R}^3)$ . That implies that the vectors of the covariant and contravariant bases belong to  $L^\infty(\omega; \mathbb{R}^3)$ .

Let the part of the boundary  $\gamma_0 \subset \partial\omega$  be of positive length. Let us define (Hilbert) spaces

$$V_N(\omega) = H_{\gamma_0}^1(\omega; \mathbb{R}^3) \times H_{\gamma_0}^1(\omega; \mathbb{R}^3) = \{(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\omega}}) \in H^1(\omega; \mathbb{R}^3)^2 : \tilde{\mathbf{v}}|_{\gamma_0} = \tilde{\boldsymbol{\omega}}|_{\gamma_0} = 0\},$$

$$V_K(\omega) = \left\{(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\omega}}) \in V_N(\omega) : \right. \\ \left. \tilde{\boldsymbol{\omega}} = \frac{1}{\sqrt{a}}((\partial_2 \tilde{\mathbf{v}} \cdot \mathbf{a}_3)\mathbf{a}_1 - (\partial_1 \tilde{\mathbf{v}} \cdot \mathbf{a}_3)\mathbf{a}_2 + \frac{1}{2}(\partial_1 \tilde{\mathbf{v}} \cdot \mathbf{a}_2 - \partial_2 \tilde{\mathbf{v}} \cdot \mathbf{a}_1)\mathbf{a}_3) \right\},$$

$$V_F(\omega) = \{(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\omega}}) \in V_N(\omega) : \partial_\beta \tilde{\mathbf{v}} + \mathbf{a}_\beta \times \tilde{\boldsymbol{\omega}} = 0, \beta = 1, 2\}$$

equipped with the norm

$$\|(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\omega}})\|_{V_N(\omega)} = \left( \|\tilde{\mathbf{v}}\|_{H^1(\omega; \mathbb{R}^3)}^2 + \|\tilde{\boldsymbol{\omega}}\|_{H^1(\omega; \mathbb{R}^3)}^2 \right)^{1/2}.$$

In the couple of functions  $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}) \in V_N(\omega)$ ,  $\tilde{\mathbf{u}}$  is the displacement vector of the middle surface of the shell, while  $\tilde{\boldsymbol{\omega}}$  is the infinitesimal rotation of the cross-section (i.e. the segment in undeformed shell perpendicular to the middle surface). The subspace  $V_K(\omega)$  corresponds to the set of unknowns in which the infinitesimal rotation  $\tilde{\boldsymbol{\omega}}$  is uniquely determined by the displacement  $\tilde{\mathbf{u}}$  such that the deformed cross-section remains perpendicular to the deformed middle surface (within the linear theory). Thus in  $V_K(\omega)$  the shell is unshearable. The set  $V_F(\omega)$  contains only inextensional displacements in  $V_K(\omega)$  (again within the linear theory). Thus in  $V_F(\omega)$  the shell is unshearable and inextensible. The subscripts of these function spaces suggest that they correspond to the Naghdi, the Koiter and the flexural shell type models. Note also that usually in the shell theories a model is given for a vectorial function which are components in the local basis  $(\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ . To emphasise the difference we use the notation  $\tilde{\mathbf{u}}$  for the physical displacement vector. It is physically natural to consider these vector displacements and actually widens the applicability of the model.

We define the bilinear forms on  $V_N(\omega) \times V_N(\omega)$  by

$$\begin{aligned} B_{ms}((\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), (\tilde{\mathbf{v}}, \tilde{\mathbf{w}})) &:= \int_{\omega} \mathbf{Q} \mathcal{C}_m \left( \mathbf{Q}^T \begin{bmatrix} \partial_1 \tilde{\mathbf{u}} + \mathbf{a}_1 \times \tilde{\boldsymbol{\omega}} & \partial_2 \tilde{\mathbf{u}} + \mathbf{a}_2 \times \tilde{\boldsymbol{\omega}} \end{bmatrix} \right) \\ &\quad \cdot \begin{bmatrix} \partial_1 \tilde{\mathbf{v}} + \mathbf{a}_1 \times \tilde{\mathbf{w}} & \partial_2 \tilde{\mathbf{v}} + \mathbf{a}_2 \times \tilde{\mathbf{w}} \end{bmatrix} \sqrt{a} d\mathbf{x}', \\ B_f((\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), (\tilde{\mathbf{v}}, \tilde{\mathbf{w}})) &:= \frac{1}{12} \int_{\omega} \mathbf{Q} \mathcal{C}_f (\mathbf{Q}^T \nabla \tilde{\boldsymbol{\omega}}) \cdot \nabla \tilde{\mathbf{w}} \sqrt{a} d\mathbf{x}'. \end{aligned}$$

The linear shell model of Naghdi type we use is following: find  $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}) \in V_N(\omega)$  such that

$$hB_{ms}((\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), (\tilde{\mathbf{v}}, \tilde{\mathbf{w}})) + h^3 B_f((\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), (\tilde{\mathbf{v}}, \tilde{\mathbf{w}})) = \int_{\omega} \tilde{\mathbf{f}} \cdot \tilde{\mathbf{v}} \sqrt{a} d\mathbf{x}', \quad (\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \in V_N(\omega). \quad (1.3.1)$$

Here,  $h$  is the shell thickness,  $\tilde{\mathbf{f}}$  is the surface force density and the elasticity tensors  $\mathcal{C}_m$ ,  $\mathcal{C}_f : \mathbb{M}^{3 \times 2} \rightarrow \mathbb{M}^{3 \times 2}$  are given by

$$\begin{aligned} \mathcal{C}_m \hat{\mathbf{C}} \cdot \hat{\mathbf{D}} &= \frac{2\lambda\mu}{\lambda + 2\mu} (\mathbf{I} \cdot \mathbf{C})(\mathbf{I} \cdot \mathbf{D}) + 2\mu \mathbf{A}_c \mathbf{C} \mathbf{A}^c \cdot \mathbf{D} + \mu \mathbf{A}^c \mathbf{c} \cdot \mathbf{d}, \\ \mathcal{C}_f \hat{\mathbf{C}} \cdot \hat{\mathbf{D}} &= a \mathcal{A} (\mathbf{J} \mathbf{C}) \cdot \mathbf{J} \mathbf{D} + a \mathcal{A}_f \mathbf{c} \cdot \mathbf{d}, \end{aligned} \quad (1.3.2)$$

where we have used the notation  $\hat{\mathbf{Q}} = \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 \end{bmatrix}$ ,  $\mathbf{Q} = \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix}$ ,  $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  and

$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{c}^T \end{bmatrix}, \quad \hat{\mathbf{D}} = \begin{bmatrix} \mathbf{D} \\ \mathbf{d}^T \end{bmatrix} \in \mathbb{M}^{3 \times 2}, \quad \mathbf{C}, \mathbf{D} \in \mathbb{M}^{2 \times 2}, \quad \mathbf{c}, \mathbf{d} \in \mathbb{R}^2.$$

We use the notation  $\hat{\cdot}$  just to compactly express the definition of elasticity tensors  $\mathcal{C}_m$  and  $\mathcal{C}_f$ . The matrix  $\mathcal{A}_f \in \mathbb{M}^{2 \times 2}$  is assumed to be positive definite and the elasticity tensor  $\mathcal{A}$  is given by

$$\mathcal{A} \mathbf{D} = \frac{2\lambda\mu}{\lambda + 2\mu} (\mathbf{A}^c \cdot \mathbf{D}) \mathbf{A}^c + 2\mu \mathbf{A}^c \mathbf{D} \mathbf{A}^c, \quad \mathbf{D} \in M^{2 \times 2}(\mathbb{R}), \quad (1.3.3)$$

where  $\lambda$  and  $\mu$  are the Lamé coefficients. Under usual assumptions  $3\lambda + 2\mu, \mu > 0$ , tensor  $\mathcal{A}$  is positive definite. On the space of symmetric matrices,  $\mathcal{A}$  coincides with the elasticity tensor that appears in the classical shell theories.

The bilinear form  $hB_{ms}((\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), (\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}))$  is related to the extensibility and shearability of the shell and measures the membrane and shear energy of the shell. The other bilinear form  $h^3 B_f((\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), (\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}))$  is related to the flexural energy. If we use space  $V_F(\omega)$  instead of  $V_N(\omega)$  in (1.3.1) we obtain the flexural shell model (see [24]), and if we use  $V_K(\omega)$  is used instead of  $V_N(\omega)$ , we obtain a Koiter type model (see [86]).

**Remark 1.3.1.** Note that when  $\boldsymbol{\theta}(\boldsymbol{x}') = (\boldsymbol{x}', 0)$  (i.e.  $\boldsymbol{\theta}$  is the parametrization of the middle surface of a plate), the model can be written in a simpler form. Due to  $\boldsymbol{a}_i = \boldsymbol{e}_i$  and  $\mathbf{A}_c = \mathbf{A}^c = \mathbf{I}$ , we have that the elasticity tensors  $\mathcal{C}_m, \mathcal{C}_f$  are given by

$$\begin{aligned}\mathcal{C}_m \hat{\mathbf{C}} \cdot \hat{\mathbf{D}} &= \mathcal{A} \mathbf{C} \cdot \mathbf{D} + \mu \mathbf{c} \cdot \mathbf{d}, \\ \mathcal{C}_f \hat{\mathbf{C}} \cdot \hat{\mathbf{D}} &= \mathcal{A} (\mathbf{J} \mathbf{C}) \cdot \mathbf{J} \mathbf{D} + a \mathcal{A}_f \mathbf{c} \cdot \mathbf{d}\end{aligned}\tag{1.3.4}$$

and

$$\mathcal{A} \mathbf{D} = \frac{2\lambda\mu}{\lambda + 2\mu} (\mathbf{I} \cdot \mathbf{D}) \mathbf{I} + 2\mu \mathbf{D}, \quad \mathbf{D} \in \mathbb{M}^{2 \times 2}.\tag{1.3.5}$$

## 2. LINEAR 3D–THIN 3D AND 3D–2D MODEL

### 2.1. INTRODUCTION

We formulate the interaction problem between 3d body and a thin layer in context of linear modelling and propose the 3d–2d linear model. Afterwards we give the main result of the chapter: the convergence theorem and all obtained limit models for all  $\alpha \geq -1$ . In Section 2.3 we do the asymptotic analysis of the 3d–thin 3d problem while in Section 2.4 we present the asymptotic analysis of the 3d–2d model.

## 2.2. DESCRIPTION OF THE PROBLEM AND THE MAIN RESULT

Let us define sets

$$\begin{aligned}
 \omega &= \langle 0, 1 \rangle \times \langle 0, 1 \rangle, \\
 \Omega^0 &= \omega \times \langle -1, 0 \rangle, \\
 \Omega^h &= \omega \times \langle 0, h \rangle, \\
 \Omega^{0+h} &= \omega \times \langle -1, h \rangle, \\
 \Gamma^s &= \omega \times \{s\}, \text{ for } s \in \mathbb{R}.
 \end{aligned} \tag{2.2.1}$$

We consider  $\overline{\Omega^{0+h}} = \overline{\Omega^0 \cup \Omega^h}$  to be an elastic body made of two materials whose properties are described by elasticity tensors  $\mathcal{C}^0$  and  $\mathcal{C}^h$ , corresponding to  $\Omega^0$  and  $\Omega^h$  (see Figure 2.1). Further we assume that the body is clamped at  $x_3 = -1$ , that contact force

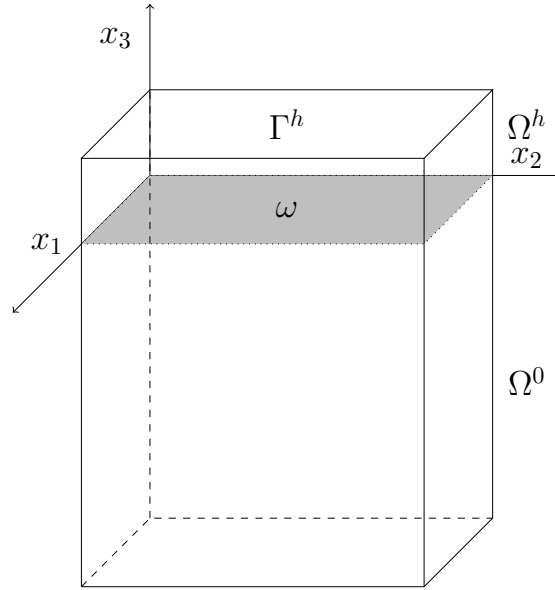


Figure 2.1: 3d elastic body.

$\mathbf{f}$  is applied at the boundary  $\Gamma^h = \omega \times \{h\}$ , and that the remaining boundary is force free. For simplicity we assume that there are no body forces applied. Since we consider linearized elasticity as a model this implies that the displacement  $\mathbf{u}^h$  corresponding to the



described problem belongs to the function space

$$V(\Omega^{0+h}) = \{\mathbf{v} \in H^1(\Omega^{0+h}) : \mathbf{v}|_{x_3=-1} = 0\}$$

and satisfies

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^h) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\Omega^h} \mathcal{C}^h \mathbf{e}(\mathbf{u}^h) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\Gamma^h} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}', \quad \mathbf{v} \in V(\Omega^{0+h}). \quad (2.2.2)$$

Here  $\mathbf{e}(\mathbf{v}) = 1/2(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$  is the symmetrized gradient and  $\mathbf{x}' = (x_1, x_2)$ .

We assume that the elasticity tensors are positive definite. Then for given  $h$  the existence and uniqueness of the solution of (2.2.2) is obtained by the use of the Lax-Milgram lemma and the Korn inequality: there is  $C_K(h) > 0$  such that

$$C_K(h) \|\mathbf{v}\|_{H^1(\Omega^{0+h}; \mathbb{R}^3)}^2 \leq \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega^{0+h}; \mathbb{R}^{3 \times 3})}^2, \quad \mathbf{v} \in V(\Omega^{0+h}). \quad (2.2.3)$$

In this chapter we consider the asymptotic behaviour of the solution  $\mathbf{u}^h$  when  $h$  tends to zero for  $\mathcal{C}^h = \frac{1}{h^{\alpha+1}} \mathcal{C}^1$ ,  $\alpha \geq -1$ , where  $\mathcal{C}^1$  is isotropic with the associated Lamé constants given by  $\lambda$  and  $\mu$ , i.e.,  $\mathcal{C}^1 \mathbf{E} = \lambda \operatorname{tr} \mathbf{E} \mathbf{I} + 2\mu \mathbf{E}$ , for  $\mathbf{E} \in \mathbb{R}^{3 \times 3}$  symmetric (the explicit elasticity coefficients of  $\mathcal{C}^0$  will never be needed). It will turn out that the limit function is independent of  $x_3$  on  $\Omega^h$  which will lead to 2d equations on  $\omega$ . As a consequence we will obtain 3d–2d models for the whole structure  $\Omega^{0+h}$ .

This kind of derivation is not new in linearized elasticity. When  $\Omega^0$  is not present this approach corresponds to the derivation of the plate model for  $\alpha = 2$ . Because of the linearity of the elasticity tensor - forcing relation this is equivalent to the scaling of the normal force with  $h^3$  for  $\alpha = 0$ , see [20]. For  $\alpha < 2$  we do not get models for all possible forces, see [2] for such models in case of the elliptic equation. Note also that here the thin plate has "support" in the elastic body and therefore can sustain the forces of order  $h^0$ . In our setting we obtain models for all  $\alpha \geq -1$  and differ five models.

Further in Section 2.4 we propose the model of interaction of the elastic body  $\Omega^0$  and the plate equation given at  $\Gamma^0$  which incorporates all models obtained by asymptotic analysis (in Section 2.3). See Theorem 2.2.1 for the precise statement. The model for the whole structure is the following: find

$$(\mathbf{u}^h, \tilde{\mathbf{u}}^h, \tilde{\omega}^h) \in V_{3d-2d} = \{(\mathbf{v}, \tilde{\mathbf{v}}, \tilde{\omega}) \in H^1(\Omega^0; \mathbb{R}^3) \times H^1(\omega; \mathbb{R}^3) \times H^1(\omega; \mathbb{R}^3) : \mathbf{v}|_{x_3=-1} = 0, \mathbf{v}|_{x_3=0} = \tilde{\mathbf{v}}\}$$

such that

$$\begin{aligned} \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^h) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \frac{1}{h^{\alpha+1}} \left( h \int_{\omega} \mathcal{C}_m (\nabla' \tilde{\mathbf{u}}^h + \mathbf{A} \tilde{\boldsymbol{\omega}}^h) \cdot (\nabla' \tilde{\mathbf{v}} + \mathbf{A} \tilde{\mathbf{w}}) d\mathbf{x}' \right. \\ \left. + \frac{h^3}{12} \int_{\omega} \mathcal{C}_f \nabla' \tilde{\boldsymbol{\omega}}^h \cdot \nabla' \tilde{\mathbf{w}} d\mathbf{x}' \right) = \int_{\omega} \tilde{\mathbf{f}} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \in V_{3d-2d}. \end{aligned} \quad (2.2.4)$$

Here  $\nabla' = (\partial_1 \partial_2)$ ,  $\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{e}_1 \times \mathbf{x} & \mathbf{e}_2 \times \mathbf{x} \end{bmatrix} \in \mathbb{R}^{3 \times 2}$  and  $\mathcal{C}_m$  and  $\mathcal{C}_f$  are elasticity tensors  $\mathcal{C}_m, \mathcal{C}_f : \mathbb{M}^{3 \times 2} \rightarrow \mathbb{M}^{3 \times 2}$  given by (1.3.4).

One can easily show, by Lax-Milgram lemma, that the solution of (2.2.4) exists and is unique. The proof is given in Theorem 2.4.1. The plate model present in (2.2.4) is obtained from the shell model of the Naghdi type for plane geometry from Chapter 1. It contains flexural, membrane and shear terms in the energy and allows all possible displacements.

In this chapter we consider asymptotics of the solution of the 3d–thin 3d problem (2.2.2) and the 3d–2d problem (2.2.4) when the thickness described by  $h$  of the top layer tends to zero. We obtain that the limits for the same  $\alpha \geq -1$  are the same. In both cases, (2.2.2) and (2.2.4), the limit models can be expressed in terms of the limits of  $\mathbf{u}^h|_{\Omega^0}$  and its trace  $\tilde{\mathbf{u}}^h = \mathbf{u}^h|_{x_3=0}$ . Thus here we formulate the convergence property only for these two functions  $\mathbf{u}^0, \tilde{\mathbf{u}}^0 = \mathbf{u}|_{x_3=0}$ . Detailed convergence results are given in Theorems 2.3.4 and 2.4.5.

We obtain five different limit models. The function spaces of the models are

$$\begin{aligned} V^I &= \{(\mathbf{v}, \tilde{\mathbf{v}}) \in H^1(\Omega^0; \mathbb{R}^3) \times L^2(\omega; \mathbb{R}^3) : \mathbf{v}^0|_{x_3=-1} = 0, \mathbf{v}^0|_{x_3=0} = \tilde{\mathbf{v}}^0\}, \\ V^{II} &= \{(\mathbf{v}, \tilde{\mathbf{v}}) \in V^I : \tilde{v}_1, \tilde{v}_2 \in H^1(\omega)\}, \\ V^{III} &= \{(\mathbf{v}, \tilde{\mathbf{v}}) \in V^{II} : \mathbf{e}'(\tilde{\mathbf{v}}) = 0\}, \\ V^{IV} &= \{(\mathbf{v}, \tilde{\mathbf{v}}) \in V^{III} : \tilde{v}_3 \in H^2(\omega)\}, \\ V^V &= \{(\mathbf{v}, \tilde{\mathbf{v}}) \in V^{IV} : D^2 \tilde{v}_3 = 0\}. \end{aligned}$$

These spaces form a monotone family

$$V^I \supseteq V^{II} \supseteq V^{III} \supseteq V^{IV} \supseteq V^V.$$

These spaces appear in a sequence of models we obtain for different values of  $\alpha \in [-1, \infty)$ . Bigger exponent corresponds to a higher  $\alpha$  which corresponds to more stiff

thin layer. In the spaces  $V^{III}$ ,  $V^{IV}$  and  $V^V$  we have that the tangential displacements satisfy  $\mathbf{e}'(\tilde{\mathbf{v}}) = 0$ , i.e, the plate is infinitesimally rigid in tangential directions. In  $V^V$  we additionally have that the transversal displacement is affine and thus the plate is infinitesimally rigid in all directions.

**Theorem 2.2.1.** Let  $\mathbf{u}_{3d}^h \in V(\Omega^{0+h})$  be the solution of (2.2.2) and let  $(\mathbf{u}_{2d}^h, \tilde{\mathbf{u}}_{2d}^h, \tilde{\mathbf{w}}_{2d}^h) \in V_{3d-2d}$  be the solution of (2.2.4). Let  $\tilde{\mathbf{u}}_{3d}^h := \mathbf{u}_{3d}^h|_{x_3=0}$ . Then for all  $\alpha \geq -1$  the families are convergent with the same limits

$$\begin{aligned} \mathbf{u}_{3d}^h|_{\Omega^0} &\rightarrow \mathbf{u}^0 && \text{strongly in } H^1(\Omega^0; \mathbb{R}^3), \\ \mathbf{u}_{2d}^h|_{\Omega^0} &\rightarrow \mathbf{u}^0 && \text{strongly in } H^1(\Omega^0; \mathbb{R}^3), \\ \tilde{\mathbf{u}}_{3d}^h &\rightarrow \tilde{\mathbf{u}}^0 && \text{strongly in } L^2(\omega; \mathbb{R}^3), \\ \tilde{\mathbf{u}}_{2d}^h &\rightarrow \tilde{\mathbf{u}}^0 && \text{strongly in } H^1(\omega; \mathbb{R}^3), \end{aligned}$$

where  $\tilde{\mathbf{u}}^0 = \mathbf{u}^0|_{x_3=0}$ . Further  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0)$  is characterized by

I) for  $-1 \leq \alpha < 0$ ,  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^I$  is the unique solution of

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^I. \quad (2.2.5)$$

II) for  $\alpha = 0$ ,  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{II}$  is the unique solution of

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\omega} \mathcal{A} \mathbf{e}'(\tilde{\mathbf{u}}^0) \cdot \mathbf{e}'(\tilde{\mathbf{v}}) d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^{II}. \quad (2.2.6)$$

III) for  $0 < \alpha < 2$ ,  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{III}$  is the unique solution of

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad \mathbf{v} \in V^{III}. \quad (2.2.7)$$

IV) for  $\alpha = 2$ ,  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{IV}$  is the unique solution of

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \frac{1}{12} \int_{\omega} \mathcal{A} D^2 \tilde{\mathbf{u}}_3^0 \cdot D^2 \tilde{\mathbf{v}}_3 d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^{IV}. \quad (2.2.8)$$

V) for  $2 < \alpha$ ,  $\mathbf{e}^0 = 0$  and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^V$  is the unique solution of

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad \mathbf{v} \in V^V. \quad (2.2.9)$$

**Remark 2.2.2.** The previous theorem is the main result of the chapter, not only because it defines limits of solutions of problems (2.2.2) and (2.2.4) for all  $\alpha \geq -1$ , but also since it states that those limits coincide for all  $\alpha \geq -1$ . In other words, the result of the above theorem also implies that

$$\|\mathbf{u}_{3d}^h|_{\Omega^0} - \mathbf{u}_{2d}^h\|_{H^1(\Omega^0; \mathbb{R}^3)} \rightarrow 0,$$

which means that in the 3d model of linearized elasticity a thin layer can be replaced by shell model of Naghdi type. Among all models mentioned in the theorem, models (2.2.5)–(2.2.9) as a drawback have that one should determine which regime (regarding the parameter  $\alpha$ ) is the best suited for the specific situation. On the other hand, model (2.2.4) approximates (2.2.2) in all regimes, and moreover has advantages over the original problem (2.2.2) which are stated in the Introduction.

In the model given by I)  $\alpha \in [-1, 0)$  the thin layer is not stiff enough so it has no influence on the limit. For  $\alpha = -1$  (meaning that the thin layer is of the same material as the remaining cube) this is in accordance with the continuous dependence of the solution on the domain. For  $\alpha = 0$ , i.e. in the case II) the membrane term of the plate appears in the total energy of the system. For all  $\alpha > 0$  this membrane energy is equal to zero. For  $\alpha \in \langle 0, 2 \rangle$ , i.e. in the case III) there is no other influence of the thin layer except of this membrane stiffening. In the case IV) ( $\alpha = 2$ ) the flexural effects appear in the total energy (the plate model), while in the case V) ( $\alpha > 2$ ) the thin layer is too stiff and behaves as a thin rigid body (no longitudinal membrane effects and no bending).

## 2.3. LIMITS OF THE 3D–THIN 3D MODEL

In this section we consider the asymptotic behavior of the solution of (2.2.2) rescaled on a reference domain for different values of  $\alpha \geq -1$ . For the limit function we will obtain the equation that uniquely determines the limit constituting the model in the case of particular  $\alpha$ . The first step in the derivation is to rescale the domain of the problem on the domain independent of  $h$ , see [19, 21] for more examples. Then we consider cases for different values of  $\alpha$ . It turns out that there are five different models as commented in Remark 2.2.2.

The uniform (with respect to  $h$ ) a priori estimates are essential for any asymptotic analysis. However, as stressed in (2.2.3) the constant in the Korn inequality depends on the domain, and thus in general it depends on  $h$ . However in the present setting it can be replaced by a constant independent of  $h$  as stated in the following lemma.

**Lemma 2.3.1.** There is  $C_K > 0$  such that for all  $h \in [0, 1]$  and for all  $\mathbf{v} \in V(\Omega^{0+h})$  one has

$$C_K \|\mathbf{v}\|_{H^1(\Omega^{0+h}; \mathbb{R}^3)}^2 \leq \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega^{0+h}; \mathbb{R}^{3 \times 3})}^2, \quad \mathbf{v} \in V(\Omega^{0+h}).$$

*Proof.* Let us define the optimal constant (the largest) for the  $h$  problem by

$$c(h) = \inf_{\substack{\mathbf{v} \in V(\Omega^{0+h}) \\ \mathbf{v} \neq 0}} \frac{\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega^{0+h}; \mathbb{R}^{3 \times 3})}^2}{\|\mathbf{v}\|_{H^1(\Omega^{0+h}; \mathbb{R}^3)}^2}.$$

In the following we prove that function  $c$  is decreasing. Let  $0 \leq h_1 < h_2 \leq 1$ . For a function  $\mathbf{v}^1 \in V(\Omega^{0+h_1})$  we define  $\mathbf{v}^2 \in V(\Omega^{0+h_2})$  by

$$\mathbf{v}^2 = \begin{cases} \mathbf{v}^1(\mathbf{x}', x_3 - h_2 + h_1) & x_3 \in \langle h_2 - h_1, 1 + h_2 \rangle, \\ 0 & x_3 \in [0, h_2 - h_1]. \end{cases}$$

Then

$$\frac{\|\mathbf{e}(\mathbf{v}^1)\|_{L^2(\Omega^{0+h_1}; \mathbb{R}^{3 \times 3})}^2}{\|\mathbf{v}^1\|_{H^1(\Omega^{0+h_1}; \mathbb{R}^3)}^2} = \frac{\|\mathbf{e}(\mathbf{v}^2)\|_{L^2(\Omega^{0+h_2}; \mathbb{R}^{3 \times 3})}^2}{\|\mathbf{v}^2\|_{H^1(\Omega^{0+h_2}; \mathbb{R}^3)}^2}$$

and thus  $c(h_1) \geq c(h_2)$ .

As a consequence  $c(h) \geq c(1) =: C_K$ . □

This lemma will provide us with the first a priori estimates after we rescale the problem on a domain independent of  $h$ . This is a natural step in order to obtain the asymptotics of the solution since then dependence on  $h$  is expressed through the coefficients in the problem. Let

$$\mathbf{R}^h : \overline{\Omega^{0+1}} \rightarrow \overline{\Omega^{0+h}}, \quad \mathbf{R}^h(\mathbf{x}', x_3) = \begin{cases} (\mathbf{x}', x_3) & x_3 \leq 0, \\ (\mathbf{x}', hx_3) & x_3 > 0, \end{cases}$$

see Figure 2.2. Now the rescaled displacement  $\mathbf{u}(h) = \mathbf{u}^h \circ \mathbf{R}^h$  belongs to  $V(\Omega^{0+1})$  and

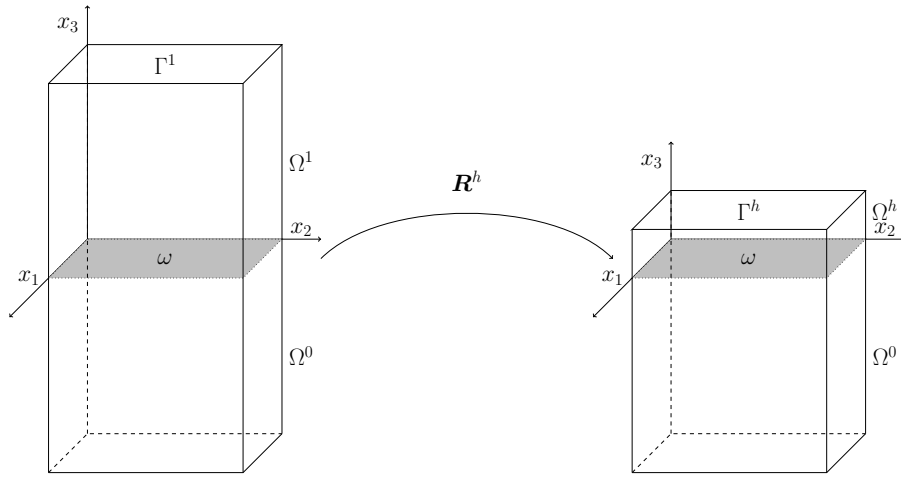


Figure 2.2: The rescaled 3d elastic body.

satisfies

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}(h)) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + h^{-\alpha} \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^h(\mathbf{u}(h)) \cdot \mathbf{e}^h(\mathbf{v}) d\mathbf{x} = \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}', \quad \mathbf{v} \in V(\Omega^{0+1}), \quad (2.3.1)$$

where

$$\mathbf{e}^h(\mathbf{v}) = \frac{1}{h} \mathbf{e}_3(\mathbf{v}) + \mathbf{e}_{12}(\mathbf{v}), \quad \mathbf{e}_3(\mathbf{v}) = \begin{bmatrix} 0 & 0 & \frac{1}{2} \partial_3 v_1 \\ 0 & 0 & \frac{1}{2} \partial_3 v_2 \\ \frac{1}{2} \partial_3 v_1 & \frac{1}{2} \partial_3 v_2 & \partial_3 v_3 \end{bmatrix},$$

$$\mathbf{e}_{12}(\mathbf{v}) = \begin{bmatrix} \partial_1 v_1 & \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) & \frac{1}{2} \partial_1 v_3 \\ \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) & \partial_2 v_2 & \frac{1}{2} \partial_2 v_3 \\ \frac{1}{2} \partial_1 v_3 & \frac{1}{2} \partial_2 v_3 & 0 \end{bmatrix}.$$

The following type of inequalities are essential for the asymptotic analysis. For similar results see [68].

**Corollary 2.3.2.** a) For all  $h \in [0, 1]$ , all  $\alpha \geq -1$  and all  $\mathbf{v} \in V(\Omega^{0+1})$  one has

$$\begin{aligned} C_K \Big( & \|\mathbf{v}\|_{H^1(\Omega^0; \mathbb{R}^3)}^2 + h\|\mathbf{v}\|_{L^2(\Omega^1; \mathbb{R}^3)}^2 \\ & + h\|\partial_1 \mathbf{v}\|_{L^2(\Omega^1; \mathbb{R}^3)}^2 + h\|\partial_2 \mathbf{v}\|_{L^2(\Omega^1; \mathbb{R}^3)}^2 + \frac{1}{h}\|\partial_3 \mathbf{v}\|_{L^2(\Omega^1; \mathbb{R}^3)}^2 \Big) \\ & \leq \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{v}) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \frac{1}{h^\alpha} \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^h(\mathbf{v}) \cdot \mathbf{e}^h(\mathbf{v}) d\mathbf{x}. \end{aligned}$$

b) For all  $h \in [0, 1]$ , all  $\alpha \geq -1$  and all  $\mathbf{v} \in V(\Omega^{0+1})$  one has

$$\|\mathbf{v}\|_{L^2(\Gamma^1; \mathbb{R}^3)}^2 \leq \|\mathbf{v}\|_{L^2(\Omega^0; \mathbb{R}^3)}^2 + \|\partial_3 \mathbf{v}\|_{L^2(\Omega^0; \mathbb{R}^3)}^2 + h\|\mathbf{v}\|_{L^2(\Omega^1; \mathbb{R}^3)}^2 + \frac{1}{h}\|\partial_3 \mathbf{v}\|_{L^2(\Omega^1; \mathbb{R}^3)}^2.$$

*Proof.* a) is a direct consequence of Lemma 2.3.1 after rescaling the domain  $\Omega^{0+h}$  to  $\Omega^{0+1}$ .

b) We estimate  $\mathbf{v}|_{\Gamma^1}$  using the Newton-Leibniz theorem and the generalized Young's inequality

$$\begin{aligned} \|\mathbf{v}\|_{L^2(\Gamma^1; \mathbb{R}^3)}^2 &= \int_{\omega} \int_{-1}^1 \partial_3(\mathbf{v}^2) dx_3 d\mathbf{x}' = \int_{\Omega^0} 2\mathbf{v} \cdot \partial_3 \mathbf{v} d\mathbf{x} + \int_{\Omega^1} 2\mathbf{v} \cdot \partial_3 \mathbf{v} d\mathbf{x} \\ &\leq \|\mathbf{v}\|_{L^2(\Omega^0; \mathbb{R}^3)}^2 + \|\partial_3 \mathbf{v}\|_{L^2(\Omega^0; \mathbb{R}^3)}^2 + h\|\mathbf{v}\|_{L^2(\Omega^1; \mathbb{R}^3)}^2 + \frac{1}{h}\|\partial_3 \mathbf{v}\|_{L^2(\Omega^1; \mathbb{R}^3)}^2. \end{aligned}$$

□

The application of this corollary for the solution of (2.3.1) is given in the following corollary.

**Corollary 2.3.3.** For all  $\alpha \geq -1$  there are sequences  $(h_k)_k \subset [0, 1]$  and  $(\mathbf{u}^k)_k \subset V(\Omega^{0+1})$  and limits  $\mathbf{u}^0$  such that  $\mathbf{u}^0|_{\Omega^0} \in V(\Omega^0)$ ,  $\mathbf{u}^0|_{\Omega^1} \in L^2(\Omega^1; \mathbb{R}^3)$ ,  $\partial_3 \mathbf{u}^0 = 0$  in  $\Omega^1$  and  $\mathbf{e}^0 \in L^2(\Omega^1; \mathbb{R}^{3 \times 3})$  such that  $h_k \rightarrow 0$  and

$$\begin{aligned} \mathbf{u}^k &\rightharpoonup \mathbf{u}^0 && \text{weakly in } H^1(\Omega^0; \mathbb{R}^3), \\ \mathbf{u}^k &\rightharpoonup \mathbf{u}^0 && \text{weakly in } L^2(\Omega^1; \mathbb{R}^3), \\ \partial_3 \mathbf{u}^k &\rightarrow 0 && \text{strongly in } L^2(\Omega^1; \mathbb{R}^3), \\ h_k^{-\alpha/2} \mathbf{e}^{h_k}(\mathbf{u}^k) &\rightharpoonup \mathbf{e}^0 && \text{weakly in } L^2(\Omega^1; \mathbb{R}^{3 \times 3}). \end{aligned} \tag{2.3.2}$$

*Proof.* Inserting  $\mathbf{u}(h)$  as a test function in (2.3.1) we obtain

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}(h)) \cdot \mathbf{e}(\mathbf{u}(h)) d\mathbf{x} + h^{-\alpha} \int_{\Omega^1} C^1 \mathbf{e}^h(\mathbf{u}(h)) \cdot \mathbf{e}^h(\mathbf{u}(h)) d\mathbf{x} = \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{u}(h) d\mathbf{x}'. \quad (2.3.3)$$

The right hand side can now be estimated using first the statement b) and then statement a) of Corollary 2.3.2

$$\begin{aligned} \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{u}(h) d\mathbf{x}' &\leq \|\mathbf{f}\|_{L^2(\Gamma^1; \mathbb{R}^3)} \|\mathbf{u}(h)\|_{L^2(\Gamma^1; \mathbb{R}^3)} \\ &\leq \|\mathbf{f}\|_{L^2(\Gamma^1; \mathbb{R}^3)} \left( \|\mathbf{u}(h)\|_{L^2(\Omega^0; \mathbb{R}^3)}^2 + \|\partial_3 \mathbf{u}(h)\|_{L^2(\Omega^0; \mathbb{R}^3)}^2 \right. \\ &\quad \left. + h \|\mathbf{u}(h)\|_{L^2(\Omega^1; \mathbb{R}^3)}^2 + \frac{1}{h} \|\partial_3 \mathbf{u}(h)\|_{L^2(\Omega^1; \mathbb{R}^3)}^2 \right)^{1/2} \\ &\leq \frac{\|\mathbf{f}\|_{L^2(\Gamma^1; \mathbb{R}^3)}}{\sqrt{C_K}} \left( \int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}(h)) \cdot \mathbf{e}(\mathbf{u}(h)) d\mathbf{x} + h^{-\alpha} \int_{\Omega^1} C^1 \mathbf{e}^h(\mathbf{u}(h)) \cdot \mathbf{e}^h(\mathbf{u}(h)) d\mathbf{x} \right)^{1/2}. \end{aligned}$$

Inserting this estimate in (2.3.3) we obtain that

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}(h)) \cdot \mathbf{e}(\mathbf{u}(h)) d\mathbf{x} + h^{-\alpha} \int_{\Omega^1} C^1 \mathbf{e}^h(\mathbf{u}(h)) \cdot \mathbf{e}^h(\mathbf{u}(h)) d\mathbf{x}$$

is bounded and then are also bounded

$$\|\mathbf{u}(h)\|_{H^1(\Omega^0; \mathbb{R}^3)}, \quad \left\| \frac{1}{\sqrt{h}} \partial_3 \mathbf{u}(h) \right\|_{L^2(\Omega^1; \mathbb{R}^3)}, \quad \|h^{-\alpha/2} \mathbf{e}^h(\mathbf{u}(h))\|_{L^2(\Omega^1; \mathbb{R}^{3 \times 3})}$$

by Corollary 2.3.2a). Further, by the Newton-Leibniz theorem and the Schwarz-Cauchy inequality we obtain

$$\begin{aligned} \mathbf{u}(h)(\mathbf{x}', x_3) &= \int_{-1}^{x_3} \partial_3 \mathbf{u}(h)(\mathbf{x}', \xi) d\xi \leq \sqrt{x_3 + 1} \left( \int_{-1}^{x_3} (\partial_3 \mathbf{u}(h)(\mathbf{x}', \xi))^2 d\xi \right)^{1/2} \\ &\leq \sqrt{2} \left( \int_{-1}^1 (\partial_3 \mathbf{u}(h))^2 d\xi \right)^{1/2}. \end{aligned}$$

After integration we obtain

$$\int_{\Omega^1} \mathbf{u}(h)^2 d\mathbf{x} \leq 2 \int_{\omega} \int_{-1}^1 (\partial_3 \mathbf{u}(h))^2 d\xi d\mathbf{x}' = 2 \int_{\Omega^{0+1}} (\partial_3 \mathbf{u}(h))^2 d\mathbf{x}.$$

Since the right hand side is uniformly bounded with respect to  $h$  we obtain that the expression  $\|\mathbf{u}(h)\|_{L^2(\Omega^1; \mathbb{R}^3)}$  is also bounded. Then by Banach–Alaoglu–Bourbaki theorem we have the stated convergences. Further, because of uniqueness of the limits we obtain that  $\partial_3 \mathbf{u}^0 = 0$  in  $\Omega^1$ .  $\square$



Since the behaviour of  $\mathbf{u}(h)$  is different on  $\Omega^0$  and  $\Omega^1$  we denote its limits by  $\mathbf{u}^0$  and  $\tilde{\mathbf{u}}^0$ , respectively. Since  $\partial_3 \tilde{\mathbf{u}}^0 = 0$  on  $\Omega^1$   $\tilde{\mathbf{u}}^0$  can be viewed as a function on  $\omega$ . We will often use the equivalence of the spaces  $\{\mathbf{v} \in L^2(\Omega^1; \mathbb{R}^3) : \partial_3 \mathbf{v} = 0\}$  and  $L^2(\omega; \mathbb{R}^3)$  and thus consider the limit of  $\mathbf{u}(h)$  as a function  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in H^1(\Omega^0; \mathbb{R}^3) \times L^2(\omega; \mathbb{R}^3)$  such that  $\mathbf{u}^0|_{x_3=0} = \tilde{\mathbf{u}}^0$ . Therefore the limit functions belongs to  $V^I$ . We collect all results on the asymptotics of  $\mathbf{u}(h)$  in the following theorem.

**Theorem 2.3.4.** Let  $\alpha \geq -1$  and let  $(\mathbf{u}(h))_h \subset V(\Omega^{0+1})$  be a family of solutions of (2.3.1). Then

$$\begin{aligned} \mathbf{u}(h) &\rightarrow \mathbf{u}^0 && \text{strongly in } H^1(\Omega^0; \mathbb{R}^3), \\ \mathbf{u}(h) &\rightarrow \mathbf{u}^0 && \text{strongly in } L^2(\Omega^1; \mathbb{R}^3), \\ \partial_3 \mathbf{u}(h) &\rightarrow 0 && \text{strongly in } L^2(\Omega^1; \mathbb{R}^3), \\ h^{-\alpha/2} \mathbf{e}^h(\mathbf{u}(h)) &\rightarrow \mathbf{e}^0 && \text{strongly in } L^2(\Omega^1; \mathbb{R}^{3 \times 3}). \end{aligned} \tag{2.3.4}$$

The limit function  $\mathbf{u}^0$  is independent of  $x_3$  in  $\Omega^1$ , i.e.  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^I$ . Furthermore  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0)$  and  $\mathbf{e}^0$  are uniquely determined by:

I) for  $-1 \leq \alpha < 0$ ,  $\mathbf{e}^0 = 0$  and  $\mathbf{u}^0$  is the unique solution of (2.2.5).

II) for  $\alpha = 0$

$$\mathbf{e}^0 = \begin{bmatrix} \partial_1 \tilde{u}_1^0 & \frac{1}{2}(\partial_1 \tilde{u}_2^0 + \partial_2 \tilde{u}_1^0) & 0 \\ \cdot & \partial_2 \tilde{u}_2^0 & 0 \\ \cdot & \cdot & -\frac{\lambda}{\lambda+2\mu}(\partial_1 \tilde{u}_1^0 + \partial_2 \tilde{u}_2^0) \end{bmatrix}.$$

and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{II}$  is the unique solution of (2.2.6). Furthermore

$$(\tilde{u}_1^h, \tilde{u}_2^h) \rightarrow (\tilde{u}_1^0, \tilde{u}_2^0) \quad \text{strongly in } H^1(\omega; \mathbb{R}^2). \tag{2.3.5}$$

III) for  $0 < \alpha < 2$ ,  $\mathbf{e}^0 = 0$  and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{III}$  is the unique solution of (2.2.7).

Furthermore (2.3.5) holds.

IV) for  $\alpha = 2$

$$\mathbf{e}^0 = \begin{bmatrix} \frac{1}{2}D^2 \tilde{u}_3^0 - x_3 D^2 \tilde{u}_3^0 & 0 \\ 0 & 0 & -\frac{\lambda}{\lambda+2\mu}(\frac{1}{2}\Delta \tilde{u}_3^0 - x_3 \Delta \tilde{u}_3^0) \end{bmatrix}.$$

and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{IV}$  is the unique solution of (2.2.8). Furthermore, the second convergence in (2.3.4) is strong in  $H^1(\Omega^1; \mathbb{R}^3)$ .

V) for  $\alpha > 2$ ,  $\mathbf{e}^0 = 0$  and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^V$  is the unique solution of (2.2.9). As in the previous case, the second convergence in (2.3.4) is also strong in  $H^1(\Omega^1; \mathbb{R}^3)$ .

### 2.3.1. 3d–thin 3d model: case $-1 \leq \alpha < 0$

We first apply Corollary 2.3.3 and extract sequences that satisfy (2.3.2). As noted above, the limit function  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0)$  belongs to  $V^I$ . Then we insert  $\mathbf{v} \in C^1(\Omega^{0+1})$  such that  $\mathbf{v}|_{x_3=-1} = 0$  and  $\mathbf{v}$  is independent of  $x_3$  on  $\Omega^1$  in (2.3.1) for this sequence and obtain

$$\begin{aligned} \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^k) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + h_k^{-\alpha/2} \int_{\Omega^1} \mathcal{C}^1(h_k^{-\alpha/2} \mathbf{e}^{h_k}(\mathbf{u}^k)) \cdot \mathbf{e}_{12}(\mathbf{v}) d\mathbf{x} \\ = \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}' = \int_{\Gamma^0} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}'. \end{aligned}$$

Since  $\alpha < 0$  one has  $-\alpha > -\alpha/2$ . Then from the last convergence in (2.3.2) it follows that  $h_k^{-\alpha} \mathbf{e}^{h_k}(\mathbf{u}^k) \rightarrow 0$  strongly in  $L^2$  and thus, after taking  $k$  to infinity we obtain

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}'. \quad (2.3.6)$$

Since chosen test functions are dense in the function space  $V(\Omega^0)$  we obtain that (2.3.6) is satisfied for  $\mathbf{v} \in V(\Omega^0)$ . This is the problem of linearized elasticity for the unit cube clamped at the bottom side ( $x_3 = -1$ ) and with non-zero contact force at the top side ( $x_3 = 0$ ). As such it has unique solution. Therefore the first three convergences in (2.3.2) hold for the whole family  $(\mathbf{u}(h))_{h>0}$ .

Let us now consider the following sequence of non-negative numbers

$$\begin{aligned} \Lambda(k) &= \int_{\Omega^0} \mathcal{C}^0(\mathbf{e}(\mathbf{u}^k) - \mathbf{e}(\mathbf{u}^0)) \cdot (\mathbf{e}(\mathbf{u}^k) - \mathbf{e}(\mathbf{u}^0)) d\mathbf{x} \\ &\quad + \int_{\Omega^1} \mathcal{C}^1(h_k^{-\alpha/2} \mathbf{e}^{h_k}(\mathbf{u}^k) - \mathbf{e}^0) \cdot (h_k^{-\alpha/2} \mathbf{e}^{h_k}(\mathbf{u}^k) - \mathbf{e}^0) d\mathbf{x}. \end{aligned}$$

Eliminating the quadratic terms using the equation (2.3.1) we obtain

$$\begin{aligned} \Lambda(k) &= \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{u}^k - 2 \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^k) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} + \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} \\ &\quad - 2 \int_{\Omega^1} \mathcal{C}^1(h_k^{-\alpha/2} \mathbf{e}^{h_k}(\mathbf{u}^k)) \cdot \mathbf{e}^0 d\mathbf{x} + \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x}. \end{aligned}$$

Next we let  $k$  to infinity using convergences (2.3.2) and in the limit obtain that  $\Lambda(k)$  converges to

$$\Lambda := \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{u}^0 - \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} - \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x} = - \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x}.$$

The last equation follows since  $\mathbf{u}^0$  satisfies the equation (2.3.6). Since  $\Lambda \geq 0$  as the limit of non-negative sequence we conclude that  $\mathbf{e}^0 = 0$  and therefore  $\Lambda = 0$ . Now we have also the uniqueness of the limit for the strain in the thin part of the domain and thus we have that all convergences in (2.3.2) hold for the whole  $h$  family. Further, since  $\Lambda = 0$  we obtain that

$$\begin{aligned} \mathbf{e}(\mathbf{u}(h)) &\rightarrow \mathbf{e}(\mathbf{u}^0) && \text{strongly in } L^2(\Omega^0; \mathbb{R}^{3 \times 3}), \\ \mathbf{e}^h(\mathbf{u}(h)) &\rightarrow 0 && \text{strongly in } L^2(\Omega^1; \mathbb{R}^{3 \times 3}). \end{aligned}$$

Then Corollary 2.3.2 implies that the convergences in (2.3.2) are all strong.

### 2.3.2. 3d–thin 3d model: case $\alpha = 0$

In this case the equation is given by find  $\mathbf{u}(h) \in V(\Omega^{0+1})$  such that

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}(h)) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^h(\mathbf{u}(h)) \cdot \mathbf{e}^h(\mathbf{v}) d\mathbf{x} = \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}', \quad \mathbf{v} \in V(\Omega^{0+1}). \quad (2.3.7)$$

From Corollary 2.3.3 we immediately know that there are sequences such that  $h_k \rightarrow 0$  and

$$\begin{aligned} \mathbf{u}^k &\rightharpoonup \mathbf{u}^0 && \text{weakly in } H^1(\Omega^0; \mathbb{R}^3), \\ \mathbf{u}^k &\rightharpoonup \mathbf{u}^0 && \text{weakly in } L^2(\Omega^1; \mathbb{R}^3), \\ \partial_3 \mathbf{u}^k &\rightarrow 0 && \text{strongly in } L^2(\Omega^1; \mathbb{R}^3), \\ \mathbf{e}^{h_k}(\mathbf{u}^k) &\rightharpoonup \mathbf{e}^0 && \text{weakly in } L^2(\Omega^1; \mathbb{R}^{3 \times 3}). \end{aligned} \quad (2.3.8)$$

and that  $\partial_3 \mathbf{u}^0 = 0$  in  $\Omega^1$ . As argued before the limit function  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^I$  a subspace of  $H^1(\Omega^0; \mathbb{R}^3) \times L^2(\omega; \mathbb{R}^3)$ . The second and the fourth convergences, by the uniqueness of the limit, imply that

$$e_{11}^0 = \partial_1 \tilde{u}_1^0, \quad e_{22}^0 = \partial_2 \tilde{u}_2^0, \quad e_{12}^0 = e_{21}^0 = \frac{1}{2}(\partial_1 \tilde{u}_2^0 + \partial_2 \tilde{u}_1^0). \quad (2.3.9)$$

Since  $e_{ij}^0$  belong to  $L^2(\Omega^1)$  we get some additional regularity on tangential displacements, namely

$$(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{II} = \{(\mathbf{v}, \tilde{\mathbf{v}}) \in V^I : \tilde{v}_1, \tilde{v}_2 \in H^1(\omega)\}.$$

Now we proceed in two steps. In the first step we multiply (2.3.7) by  $h_k$  and let  $k$  to infinity. We obtain

$$\int_{\Omega^1} (\mathcal{C}^1 \mathbf{e}^0) \cdot \mathbf{e}_3(\mathbf{v}) = 0, \quad \mathbf{v} \in V(\Omega^{0+1}).$$

This implies  $(\mathcal{C}^1 \mathbf{e}^0)_{i,3} = (\mathcal{C}^1 \mathbf{e}^0)_{3,i} = 0$  for  $i = 1, 2, 3$  and thus

$$e_{13}^0 = e_{23}^0 = e_{31}^0 = e_{32}^0 = 0, \quad e_{33}^0 = -\frac{\lambda}{\lambda + 2\mu}(e_{11}^0 + e_{22}^0) = -\frac{\lambda}{\lambda + 2\mu}(\partial_1 u_1^0 + \partial_2 u_2^0). \quad (2.3.10)$$

In the second step we insert in (2.3.7) test functions from  $V(\Omega^{0+1})$  independent of  $x_3$  on  $\Omega^1$  and take the limit when  $k$  tends to infinity. Due to (2.3.8) we obtain

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}_{12}(\mathbf{v}) d\mathbf{x} = \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}' = \int_{\Gamma^0} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}'.$$

Now (2.3.10) implies that

$$\mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}_{12}(\mathbf{v}) = \mathcal{A} \mathbf{e}'(\mathbf{u}^0) \cdot \mathbf{e}'(\mathbf{v}),$$

where  $\mathbf{e}'(\mathbf{v}) = 1/2(\nabla'(v_1, v_2) + \nabla'(\mathbf{v}_1, \mathbf{v}_2)^T)$ , where in  $\nabla'$  only derivative with respect to variables  $x_1$  and  $x_2$  are taken and  $\mathcal{A}$  is given in (1.3.5). Since the function integrated over  $\Omega^1$  does not depend on  $x_3$  this integral can be replaced by the integral over  $\omega$ . Thus we obtain that the limit function  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{II}$  satisfies

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\omega} \mathcal{A} \mathbf{e}'(\tilde{\mathbf{u}}^0) \cdot \mathbf{e}'(\tilde{\mathbf{v}}) d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^{II}. \quad (2.3.11)$$

Due to the Korn inequality on  $\Omega^0$ , the trace theorem for  $(\mathbf{v}, \tilde{\mathbf{v}}) \in V^{II}$  and the Korn inequality on  $\omega$  we have

$$\begin{aligned} & \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega^0; \mathbb{R}^{3 \times 3})}^2 + \|\mathbf{e}'(\tilde{\mathbf{v}})\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \\ & \geq c(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|\mathbf{v}\|_{L^2(\Gamma^0; \mathbb{R}^3)}^2) + \|\mathbf{e}'(\tilde{\mathbf{v}})\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \\ & \geq c\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + c\|\tilde{\mathbf{v}}\|_{L^2(\omega; \mathbb{R}^3)}^2 + \|\mathbf{e}'(\tilde{\mathbf{v}})\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \\ & \geq c(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|(\tilde{v}_1, \tilde{v}_2)\|_{H^1(\omega; \mathbb{R}^2)}^2 + \|\tilde{v}_3\|_{L^2(\omega)}^2). \end{aligned} \quad (2.3.12)$$

This implies coercivity of the form on the left hand side on  $V^{II}$  with respect to the natural norm on  $L^2(\Omega^0; \mathbb{R}^3) \times H^1(\omega) \times H^1(\omega) \times L^2(\omega)$  and thus existence and uniqueness of the solution of (2.3.11) in  $V^{II}$ . Therefore  $\mathbf{e}^0$  is also unique and given by

$$\mathbf{e}^0 = \begin{bmatrix} \partial_1 \tilde{u}_1^0 & \frac{1}{2}(\partial_1 \tilde{u}_2^0 + \partial_2 \tilde{u}_1^0) & 0 \\ \cdot & \partial_2 \tilde{u}_2^0 & 0 \\ \cdot & \cdot & -\frac{\lambda}{\lambda + 2\mu}(\partial_1 \tilde{u}_1^0 + \partial_2 \tilde{u}_2^0) \end{bmatrix}.$$

Thus the whole  $h$ -families, corresponding to (2.3.8) converge.

To obtain the strong convergence we argue as before. We define

$$\begin{aligned} \Lambda(h) = \int_{\Omega^0} \mathcal{C}^0(\mathbf{e}(\mathbf{u}(h)) - \mathbf{e}(\mathbf{u}^0)) \cdot (\mathbf{e}(\mathbf{u}(h)) - \mathbf{e}(\mathbf{u}^0)) d\mathbf{x} + \\ \int_{\Omega^1} \mathcal{C}^1(\mathbf{e}^h(\mathbf{u}(h)) - \mathbf{e}^0) \cdot (\mathbf{e}^h(\mathbf{u}(h)) - \mathbf{e}^0) d\mathbf{x}. \end{aligned}$$

We eliminate the quadratic terms using the equation (2.3.1) and take the limit as  $h$  tends to zero. We obtain

$$\begin{aligned} \Lambda(h) \rightarrow \Lambda := \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{u}^0 - \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} - \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x} \\ = \int_{\omega} \mathcal{A} \mathbf{e}'(\tilde{\mathbf{u}}^0) \cdot \mathbf{e}'(\tilde{\mathbf{u}}^0) d\mathbf{x}' - \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x}. \end{aligned}$$

A simple calculation for obtained  $\mathbf{e}^0$  implies that  $\Lambda = 0$ . Thus we have the strong convergence of the term  $\mathbf{e}(\mathbf{u}(h))$  on  $\Omega^0$  and  $\mathbf{e}^h(\mathbf{u}(h))$  on  $\Omega^1$  and thus all convergences in (2.3.8) are strong.

### 2.3.3. 3d–thin 3d model: case $0 < \alpha < 2$

Since  $\alpha > 0$  from the last convergence in (2.3.2) we conclude that  $\mathbf{e}^{h_k}(\mathbf{u}^k) \rightarrow 0$  strongly in  $L^2(\Omega^1; \mathbb{R}^{3 \times 3})$ . Arguing as in the case  $\alpha = 0$  we conclude that

$$\partial_1 u_1^0 = \partial_2 u_2^0 = \frac{1}{2}(\partial_1 u_2^0 + \partial_2 u_1^0) = 0 \quad \text{in } \Omega^1,$$

i.e.

$$\mathbf{e}'(\tilde{\mathbf{u}}^0) = 0 \quad \text{in } \Omega^1. \quad (2.3.13)$$

Additionally, because of (2.3.13) we get some additional information on tangential displacements

$$(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{III} = \{(\mathbf{v}, \tilde{\mathbf{v}}) \in V^{II} : \mathbf{e}'(\tilde{\mathbf{v}}) = 0\}.$$

Let  $\mathbf{v} \in V(\Omega^{0+1})$  be such that  $\partial_3 \mathbf{v} = 0$ ,  $\mathbf{e}'(\mathbf{v}) = 0$  in  $\Omega^1$  and  $\tilde{v}_3$  smooth. Let

$$\mathbf{v}^1 = 0 \quad \text{in } \Omega^0, \quad \mathbf{v}^1 = -x_3 \begin{bmatrix} \partial_1 v_3 \\ \partial_2 v_3 \\ 0 \end{bmatrix} \quad \text{in } \Omega^0. \quad (2.3.14)$$

Then in  $\Omega^1$

$$\mathbf{e}^h(\mathbf{v} + h\mathbf{v}^1) = \frac{1}{h}\mathbf{e}_3(\mathbf{v}) + \mathbf{e}_{12}(\mathbf{v}) + \mathbf{e}_3(\mathbf{v}^1) + h\mathbf{e}_{12}(\mathbf{v}^1) = -hx_3 \begin{bmatrix} D^2(v_3) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (2.3.15)$$

We insert  $\mathbf{v} + h\mathbf{v}^1$  in (2.3.1) and get

$$\begin{aligned} \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}(h_k)) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\Omega^1} \mathcal{C}^1 \frac{1}{h_k^{\alpha/2}} \mathbf{e}^{h_k}(\mathbf{u}(h_k)) \cdot (h_k^{(2-\alpha)/2}) x_3 \begin{bmatrix} D^2(v_3) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} d\mathbf{x} \\ = \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}'. \end{aligned}$$

Since  $0 < \alpha < 2$  by letting  $k$  to infinity we obtain that  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{III}$  satisfies

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\Gamma^0} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^{III}, \quad (2.3.16)$$

since used test functions  $\mathbf{v}$  are dense in  $V^{III}$ . The estimate from (2.3.12) implies that on  $V^{III}$  one has

$$\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega^0; \mathbb{R}^{3 \times 3})}^2 \geq c(\|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + \|(\tilde{v}_1, \tilde{v}_2)\|_{H^1(\omega; \mathbb{R}^2)}^2 + \|\tilde{v}_3\|_{L^2(\omega)}^2). \quad (2.3.17)$$

This implies coercivity of the form on the left hand side with respect to the natural norm on  $L^2(\Omega^0; \mathbb{R}^3) \times H^1(\omega) \times H^1(\omega) \times L^2(\omega)$  and thus existence and uniqueness of the solution of (2.3.16) in  $V^{III}$ . Therefore the whole  $h$ -family  $(\mathbf{u}(h))_h$  converges as in the first three convergences in (2.3.8).

The same arguments as in the case  $\alpha \in [-1, 0)$  leads to the same conclusion, that  $\mathbf{e}^0 = 0$  and that all convergences in (2.3.2) hold for the whole  $h$  family and that they are all strong.

#### 2.3.4. 3d–thin 3d model: case $\alpha = 2$

In this case the equation is given by: find  $\mathbf{u}(h) \in V(\Omega^{0+1})$  such that

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}(h)) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \frac{1}{h^2} \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^h(\mathbf{u}(h)) \cdot \mathbf{e}^h(\mathbf{v}) d\mathbf{x} = \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}', \quad \mathbf{v} \in V(\Omega^{0+1}). \quad (2.3.18)$$

Since we have  $1/h^2$  in front of the thin part, now we have better a priori estimate than in Corollary 2.3.2.

**Lemma 2.3.5.** There is  $C_K \geq 0$  such that for all  $h \in [0, 1]$  and all  $\mathbf{v} \in V(\Omega^{0+1})$  one has

$$C_K \|\mathbf{v}\|_{H^1(\Omega^{0+1}; \mathbb{R}^3)}^2 \leq \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{v}) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \frac{1}{h^2} \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^h(\mathbf{v}) \cdot \mathbf{e}^h(\mathbf{v}) d\mathbf{x}.$$

*Proof.* If we assume the opposite there exist a sequence  $(\mathbf{v}^k) \subset V(\Omega^{0+1})$  and a limit function  $\mathbf{v} \in V(\Omega^{0+1})$  such that  $\|\mathbf{v}\|_{H^1(\Omega^{0+1}; \mathbb{R}^3)} = 1$  and

$$\begin{aligned} \mathbf{v}^k &\rightharpoonup \mathbf{v} && \text{weakly in } H^1(\Omega^{0+1}; \mathbb{R}^3), \\ \mathbf{e}(\mathbf{v}^k) &\rightarrow 0 && \text{strongly in } H^1(\Omega^0; \mathbb{R}^{3 \times 3}), \\ \frac{1}{h_k} \mathbf{e}^{h_k}(\mathbf{v}^k) &\rightarrow 0 && \text{strongly in } H^1(\Omega^1; \mathbb{R}^{3 \times 3}). \end{aligned} \quad (2.3.19)$$

From the first two convergences and the classical Korn inequality we have  $\mathbf{v} = 0$  in  $\Omega^0$  and

$$\mathbf{v}^k \rightarrow 0 \quad \text{strongly in } H^1(\Omega^0; \mathbb{R}^3).$$

From Corollary 2.3.2a) we obtain that  $\partial_3 \mathbf{v}^k \rightarrow 0$  strongly in  $L^2(\Omega^1; \mathbb{R}^3)$ . Next we apply the classical Korn inequality on  $V(\Omega^{0+1})$ , but for the sequence  $(v_1^k, v_2^k, h_k v_3^k)$ . We obtain

$$\begin{aligned} C_K \|(v_1^k, v_2^k, h_k v_3^k)\|_{H^1(\Omega^{0+1}; \mathbb{R}^3)}^2 &\leq \|\mathbf{e}(v_1^k, v_2^k, h_k v_3^k)\|_{L^2(\Omega^0; \mathbb{R}^{3 \times 3})}^2 + \|\mathbf{e}(v_1^k, v_2^k, h_k v_3^k)\|_{L^2(\Omega^1; \mathbb{R}^{3 \times 3})}^2 \\ &\leq \|\mathbf{v}^k\|_{H^1(\Omega^0; \mathbb{R}^3)}^2 + \|\mathbf{e}'(v_1^k, v_2^k)\|_{L^2(\Omega^1; \mathbb{R}^{2 \times 2})}^2 \\ &\quad + \|h_k \partial_1 v_3^k + \partial_3 v_1^k\|_{L^2(\Omega^1)}^2 + \|h_k \partial_2 v_3^k + \partial_3 v_2^k\|_{L^2(\Omega^1)}^2 + \|h_k \partial_3 v_3^k\|_{L^2(\Omega^1)}^2 \\ &\leq \|\mathbf{v}^k\|_{H^1(\Omega^0; \mathbb{R}^3)}^2 + \|\mathbf{e}^{h_k}(\mathbf{v}^k)\|_{L^2(\Omega^1; \mathbb{R}^{3 \times 3})}^2. \end{aligned}$$

Thus we also have that  $v_1^k, v_2^k \rightarrow 0$  strongly in  $H^1(\Omega^1)$ . Thus  $v_1 = v_2 = 0$  in  $\Omega^1$  and we only have to prove that

$$v_3^k \rightarrow 0 \quad \text{strongly in } H^1(\Omega^1). \quad (2.3.20)$$

Since  $\partial_3 \mathbf{v}^k \rightarrow 0$  strongly in  $L^2(\Omega^1)$  we know that  $\partial_3 v_3 = 0$  in  $\Omega^1$ . Since  $\mathbf{v}|_{x_3=0} = 0$  as a trace of  $H^1$  function on  $\Omega^0$  we obtain that  $v_3 = 0$ . Now we are missing the following two strong convergences

$$\partial_1 v_3^k \rightarrow 0, \quad \partial_2 v_3^k \rightarrow 0 \quad \text{strongly in } L^2(\Omega^1). \quad (2.3.21)$$

We now prove the first convergence by use of the Lions lemma ( $f \in L^2$  is isomorphic to  $f, \nabla' f \in H^{-1}$ ). This part of the proof is the same as in the case of the plate equation.

From the third convergence in (2.3.19) we have

$$\begin{aligned}\partial_{11}v_3^k &= 2h_k\partial_1\mathbf{e}^{h_k}(\mathbf{v}^k)_{13} - \partial_3\mathbf{e}^{h_k}(\mathbf{v}^k)_{11}, \\ \partial_{21}v_3^k &= h_k\partial_2\mathbf{e}^{h_k}(\mathbf{v}^k)_{13} + h_k\partial_1\mathbf{e}^{h_k}(\mathbf{v}^k)_{23} - \partial_3\mathbf{e}^{h_k}(\mathbf{v}^k)_{12}, \\ \partial_{31}v_3^k &= h_k^2\partial_1\mathbf{e}^{h_k}(\mathbf{v}^k)_{13}.\end{aligned}$$

Since the all terms on the right hand side converge strongly in  $H^{-1}(\Omega^1)$  to 0 we have the same convergence for  $\nabla\partial_1v_3^k$ . Further from the first convergence in (2.3.19) we obtain that  $\partial_1v_3^k$  converges strongly in  $H^{-1}(\Omega)$ . Thus by the Lions lemma we have the strong convergence of  $\partial_1v_3^k$  in  $L^2(\Omega^1)$ . The same arguments proves the strong convergence of  $\partial_2v_3^k$  in  $L^2$ . As a consequence we obtain that the first convergence in (2.3.19) is also strong. Since we already have that the limit  $\mathbf{v} = 0$  we arrive at contradiction with  $\|\mathbf{v}\|_{H^1(\Omega^{0+1};\mathbb{R}^3)} = 1$ .  $\square$

From Corollary 2.3.3 and the previous lemma we immediately know that there are sequences such that  $h_k \rightarrow 0$  and

$$\begin{aligned}\mathbf{u}^k &\rightharpoonup \mathbf{u}^0 \quad \text{weakly in } H^1(\Omega^0; \mathbb{R}^3), \\ \partial_3\mathbf{u}^k &\rightarrow 0 \quad \text{strongly in } L^2(\Omega^1; \mathbb{R}^3), \\ \mathbf{u}^k &\rightharpoonup \mathbf{u}^0 \quad \text{weakly in } H^1(\Omega^1; \mathbb{R}^3), \\ \frac{1}{h_k}\mathbf{e}^{h_k}(\mathbf{u}^k) &\rightharpoonup \mathbf{e}^0 \quad \text{weakly in } L^2(\Omega^1; \mathbb{R}^{3 \times 3}).\end{aligned}\tag{2.3.22}$$

and that  $\partial_3\mathbf{u}^0 = 0$  in  $\Omega^1$ .

The following theorem relates  $\mathbf{u}^0$  and  $\mathbf{e}^0$  on  $\Omega^1$ . It is classical in the plate derivations and is the most delicate part of the derivation of the plate model.

**Theorem 2.3.6** ([21, Theorem 5.2-2]). Let the family  $(\mathbf{w}(h))_{h>0} \subset H^1(\Omega^1)$  satisfies

$$\begin{aligned}\mathbf{w}(h) &\rightharpoonup \mathbf{w}^0 \text{ weakly in } H^1(\Omega^1; \mathbb{R}^3), \\ \frac{1}{h}\mathbf{e}^h(\mathbf{w}(h)) &\rightharpoonup \mathbf{e}^0 \text{ weakly in } L^2(\Omega^1; \mathbb{R}^{3 \times 3})\end{aligned}$$

as  $h \rightarrow 0$ . Then the limit function  $\mathbf{w}^0$  is independent of transverse variable  $x_3$ , belongs to  $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$  and satisfies the following conditions

$$\mathbf{e}'(\mathbf{w}^0) = 0 \text{ and } \frac{\partial e_{\beta_1\beta_2}^0}{\partial x_3} = -\partial_{\beta_1\beta_2}w_3, \quad \beta_1, \beta_2 \in \{1, 2\}.$$



Applying this theorem on  $\mathbf{u}(h)$  we conclude that  $\mathbf{e}'(\tilde{\mathbf{u}}^0) = 0$  and that  $\frac{\partial e_{\beta_1\beta_2}^0}{\partial x_3} = -\partial_{\beta_1\beta_2}\tilde{u}_3^0$ ,  $\beta_1, \beta_2 \in \{1, 2\}$ . Thus the limit belongs to

$$(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{IV} = \{(\mathbf{v}, \tilde{\mathbf{v}}) \in V^{III} : \tilde{v}_3 \in H^2(\omega)\}.$$

Now we proceed in three steps. In the first step we multiply (2.3.18) for  $h_k$  by  $h_k^2$  and let  $k$  to infinity. We obtain, as in the step  $\alpha = 0$ ,

$$\int_{\Omega^1} (\mathcal{C}^1 \mathbf{e}^0) \cdot \mathbf{e}_3(\mathbf{v}) = 0, \quad \mathbf{v} \in V(\Omega^{0+1}).$$

This implies  $(\mathcal{C}^1 \mathbf{e}^0)_{i,3} = (\mathcal{C}^1 \mathbf{e}^0)_{3,i} = 0$  for  $i = 1, 2, 3$  and thus

$$e_{13}^0 = e_{23}^0 = 0, \quad e_{33}^0 = -\frac{\lambda}{\lambda + 2\mu}(e_{11}^0 + e_{22}^0).$$

Thus we characterize  $\mathbf{e}^0$  up to a 2 by 2 matrix  $\tilde{\mathbf{e}}$  function independent of  $x_3$ :

$$\mathbf{e}^0(\mathbf{x}) = \begin{bmatrix} \tilde{\mathbf{e}}(\mathbf{x}') - x_3 D^2 \tilde{u}_3^0(\mathbf{x}') & 0 \\ 0 & 0 \\ 0 & 0 & -\frac{\lambda}{\lambda + 2\mu}(\text{tr } \tilde{\mathbf{e}}(\mathbf{x}') - x_3 \Delta \tilde{u}_3^0(\mathbf{x}')) \end{bmatrix}.$$

In the second step we gain further information about  $\tilde{\mathbf{e}}$ . We insert  $\mathbf{v} \in V(\Omega^{0+1})$  such that  $\partial_3 \mathbf{v} = 0$  in  $\Omega^1$ , i.e.,  $(\mathbf{v}, \tilde{\mathbf{v}}) \in V^{II}$  in (2.3.18), multiply the equation by  $h_k$ , and let  $k$  tends to infinity. We obtain that

$$\int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}_{12}(\tilde{\mathbf{v}}) d\mathbf{x} = 0.$$

From the form of  $\mathbf{e}^0$  this implies

$$\int_{\Omega^1} \mathcal{A}(\tilde{\mathbf{e}} - x_3 D^2 \tilde{u}_3^0) \cdot \mathbf{e}'(\tilde{\mathbf{v}}) d\mathbf{x} = 0.$$

Thus we obtain

$$\int_{\omega} \mathcal{A} \tilde{\mathbf{e}} \cdot \mathbf{e}'(\tilde{\mathbf{v}}) d\mathbf{x}' = \frac{1}{2} \int_{\omega} \mathcal{A}(D^2 \tilde{u}_3^0) \cdot \mathbf{e}'(\tilde{\mathbf{v}}) d\mathbf{x}'. \quad (2.3.23)$$

In the third step we insert in (2.3.18) test functions of the form  $\mathbf{v} + h\mathbf{v}^1$ , where  $(\mathbf{v}, \tilde{\mathbf{v}}) \in V^{IV}$  and  $\mathbf{v}^1$  is given in (2.3.14). Then, according to (2.3.15) the obtained equation is given by

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}(h_k)) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\Omega^1} \mathcal{C}^1 \frac{1}{h_k} \mathbf{e}^{h_k}(\mathbf{u}(h_k)) \cdot \begin{bmatrix} x_3 D^2(v_3) & 0 \\ 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d\mathbf{x} = \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}'.$$

Taking the limit as  $k$  tends to infinity from this equation we obtain

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \begin{bmatrix} -x_3 D^2(\tilde{v}_3) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} d\mathbf{x} = \int_{\Gamma^0} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}'. \quad (2.3.24)$$

Now the form of  $\mathbf{e}^0$  implies that

$$\mathcal{C}^1 \mathbf{e}^0 \cdot \begin{bmatrix} -x_3 D^2(\tilde{v}_3) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = x_3^2 \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{v}_3 - x_3 \mathcal{A} \tilde{\mathbf{e}} \cdot D^2 \tilde{v}_3.$$

We insert this in (2.3.24) and get

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\omega} \frac{1}{3} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{v}_3 - \frac{1}{2} \mathcal{A} \tilde{\mathbf{e}} \cdot D^2 \tilde{v}_3 d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}'.$$

Since the function  $D^2 \tilde{v}_3$  can be written in the form of symmetrized gradient using (2.3.23) we get that the limit function  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{IV}$  and satisfies

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \frac{1}{12} \int_{\omega} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{v}_3 d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^{IV}. \quad (2.3.25)$$

Using (2.3.17) we obtain

$$\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega^0; \mathbb{R}^{3 \times 3})}^2 + \|D^2 \tilde{v}_3\|_{L^2(\omega)}^2 \geq c \|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + c \|(\tilde{v}_1, \tilde{v}_2)\|_{H^1(\omega; \mathbb{R}^2)}^2 + c \|\tilde{v}_3\|_{H^2(\omega)}^2. \quad (2.3.26)$$

Thus the form on the left hand side of (2.3.25) is elliptic with respect to the standard norm on  $V^{IV}$  given by  $L^2(\Omega^0; \mathbb{R}^3) \times H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ . Thus we can apply the Lax-Milgram lemma to obtain existence and uniqueness of the solution of (2.3.25). Hence the whole family  $\mathbf{u}(h)$  converges to the same limit  $\mathbf{u}^0$ .

Next we identify  $\tilde{\mathbf{e}}$  and thus the limit  $\mathbf{e}^0$  and prove the strong convergences in (2.3.22).

Let us substitute

$$\tilde{\mathbf{e}} = \frac{1}{2} D^2 \tilde{u}_3^0 + \tilde{\tilde{\mathbf{e}}}.$$

Then from (2.3.23)  $\tilde{\tilde{\mathbf{e}}}$  satisfies

$$\int_{\omega} \mathcal{A} \tilde{\tilde{\mathbf{e}}} \cdot \mathbf{e}'(\tilde{\mathbf{v}}) d\mathbf{x} = 0 \quad (2.3.27)$$

for all  $\tilde{\mathbf{v}} \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ .

To obtain the strong convergence we argue as before. We define

$$\begin{aligned}\Lambda(k) &= \int_{\Omega^0} \mathcal{C}^0(\mathbf{e}(\mathbf{u}(h_k)) - \mathbf{e}(\mathbf{u}^0)) \cdot (\mathbf{e}(\mathbf{u}(h_k)) - \mathbf{e}(\mathbf{u}^0)) d\mathbf{x} \\ &\quad + \int_{\Omega^1} \mathcal{C}^1 \left( \frac{1}{h_k} \mathbf{e}^{h_k}(\mathbf{u}(h_k)) - \mathbf{e}^0 \right) \cdot \left( \frac{1}{h_k} \mathbf{e}^{h_k}(\mathbf{u}(h_k)) - \mathbf{e}^0 \right) d\mathbf{x}.\end{aligned}$$

As before we eliminate the quadratic terms using the equation (2.3.18) and take the limit as  $k$  tends to infinity. After taking into account (2.3.25) we obtain

$$\begin{aligned}\Lambda(k) \rightarrow \Lambda &:= \int_{\Gamma^1} \mathbf{f} \cdot \mathbf{u}^0 - \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} - \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x} \\ &= \frac{1}{12} \int_{\omega} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{u}_3^0 d\mathbf{x}' - \int_{\Omega^1} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x}.\end{aligned}$$

Next we compute

$$\begin{aligned}\int_{\Omega} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x} &= \int_{\Omega} \mathcal{A}(\tilde{\mathbf{e}} - x_3 D^2 \tilde{u}_3^0) \cdot (\tilde{\mathbf{e}} - x_3 D^2 \tilde{u}_3^0) d\mathbf{x} \\ &= \int_{\omega} \mathcal{A} \tilde{\mathbf{e}} \cdot \tilde{\mathbf{e}} d\mathbf{x}' - \int_{\omega} \mathcal{A} \tilde{\mathbf{e}} \cdot D^2 \tilde{u}_3^0 d\mathbf{x}' + \frac{1}{3} \int_{\omega} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{u}_3^0 d\mathbf{x}'.\end{aligned}$$

Then, we substitute  $\tilde{\mathbf{e}}$  by  $\tilde{\tilde{\mathbf{e}}}$  and obtain

$$\begin{aligned}\int_{\Omega} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x} &= \int_{\omega} \mathcal{A} \tilde{\tilde{\mathbf{e}}} \cdot \tilde{\tilde{\mathbf{e}}} d\mathbf{x}' + \int_{\omega} \mathcal{A} \tilde{\tilde{\mathbf{e}}} \cdot D^2 \tilde{u}_3^0 d\mathbf{x}' + \frac{1}{4} \int_{\omega} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{u}_3^0 d\mathbf{x} \\ &\quad - \int_{\omega} \mathcal{A} \tilde{\tilde{\mathbf{e}}} \cdot D^2 \tilde{u}_3^0 d\mathbf{x}' - \frac{1}{2} \int_{\omega} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{u}_3^0 d\mathbf{x}' + \frac{1}{3} \int_{\omega} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{u}_3^0 d\mathbf{x}'.\end{aligned}$$

Using (2.3.27) we obtain

$$\int_{\Omega} \mathcal{C}^1 \mathbf{e}^0 \cdot \mathbf{e}^0 d\mathbf{x} = \int_{\omega} \mathcal{A} \tilde{\tilde{\mathbf{e}}} \cdot \tilde{\tilde{\mathbf{e}}} d\mathbf{x}' + \frac{1}{12} \int_{\omega} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{u}_3^0 d\mathbf{x}.$$

Then we insert this expression in formula for  $\Lambda$  and obtain

$$\Lambda(k) \rightarrow \Lambda := - \int_{\omega} \mathcal{A} \tilde{\tilde{\mathbf{e}}} \cdot \tilde{\tilde{\mathbf{e}}} d\mathbf{x}'.$$

Since  $\Lambda(k) \geq 0$  this implies that  $\Lambda = 0$  and therefore  $\tilde{\tilde{\mathbf{e}}} = 0$ . Therefore  $\mathbf{e}^0$  is uniquely determined and we obtain the convergence of the whole families in (2.3.22). Further we obtain the strong convergence of the term  $\mathbf{e}(\mathbf{u}(h))$  on  $\Omega^0$  and  $\frac{1}{h} \mathbf{e}^h(\mathbf{u}(h))$  on  $\Omega^1$  and thus all convergences in (2.3.22) are strong.

### 2.3.5. 3d–thin 3d model: case $\alpha > 2$

From the last convergence in (2.3.2), since  $\alpha > 2$ , we conclude that

$$\frac{1}{h_k} \mathbf{e}^{h_k}(\mathbf{u}^k) \rightarrow 0 \quad \text{strongly in } L^2(\Omega^1; \mathbb{R}^{3 \times 3}).$$

Therefore we can apply Lemma 2.3.5 and obtain that (2.3.22) hold. Thus, as in the case  $\alpha = 2$ , we can apply Theorem 2.3.6 with  $\mathbf{e}^0 = 0$  and conclude that

$$\mathbf{e}'(\tilde{\mathbf{u}}^0) = 0, \quad D^2 \tilde{\mathbf{u}}_3^0 = 0.$$

Thus the limit belongs to

$$(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^V = \{(\mathbf{v}, \tilde{\mathbf{v}}) \in V^{IV} : D^2 \tilde{\mathbf{v}}_3 = 0\}.$$

Let  $(\mathbf{v}, \tilde{\mathbf{v}}) \in V^V$  and choose test functions in the form  $\mathbf{v}(h) = \mathbf{v} + h\mathbf{v}^1$  as defined in (2.3.14). Thus according to (2.3.15) one has that  $\mathbf{e}^h(\mathbf{v}(h)) = 0$ . Therefore for such test functions in (2.3.1) the integral over  $\Omega^1$  disappears. Then we let  $k$  to infinity and easily obtain that the limit function  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^V$  satisfies

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^V. \quad (2.3.28)$$

From (2.3.26) we obtain that on  $V^V$  one has

$$\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega^0; \mathbb{R}^{3 \times 3})}^2 \geq c \|\mathbf{v}\|_{H^1(\Omega; \mathbb{R}^3)}^2 + c \|(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2)\|_{H^1(\omega; \mathbb{R}^2)}^2 + c \|\tilde{\mathbf{v}}_3\|_{H^2(\omega)}^2$$

which implies ellipticity of the form on the left hand side of (2.3.28) with respect to the norm on  $V^V$  given by the standard norm on  $L^2(\Omega^0; \mathbb{R}^3) \times H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ . The existence and uniqueness of solution of (2.3.28) follows. Again, convergence of the whole family  $\mathbf{u}(h)$  follows by uniqueness.

Arguing as in the cases  $\alpha \in [-1, 0)$  and  $\alpha \in \langle 0, 2 \rangle$  we obtain that  $\mathbf{e}^0 = 0$ , that the convergence of the whole family  $h^{-\alpha/2} \mathbf{e}^h(\mathbf{u}(h))$  holds and that all convergences in (2.3.2) and (2.3.22) are strong.

## 2.4. LIMITS OF THE 3D–2D MODEL

In this section we start from the model in which a twodimensional structure is attached to the unit cube at  $x_3 = 0$ . At the unit cube we consider the linearized elasticity with fixed material, while for the plate we use the Naghdi type shell model from the Chapter 1 with plane geometry. We attach the plate to the cube by setting that the plate displacement is equal to the displacement of the cube at the top (for  $x_3 = 0$ ), similarly as it was done in Section 2.3 in different models. We assume the thickness of the plate  $h$  is a small, i.e.  $h \ll 1$ , and that the elasticity coefficients of the plate are related to the thickness as before, i.e. of order  $1/h^{\alpha+1}$ ,  $\alpha \geq -1$ . Then we show that this 3d–2d model in each regime asymptotically with respect to  $h$  behaves in the same way the 3d–thin 3d model from Section 2.3 behaves. Thus instead of models from the previous section we can use this model not paying attention on the orders of ratio of the elasticity coefficients.

**Theorem 2.4.1.** The problem (2.2.4) has unique solution.

The statement of the theorem follows from the Lax–Milgram lemma. The key part is the  $V_{3d-2d}$  ellipticity of the form in the left hand side of (2.2.4). It follows from the following estimate.

**Lemma 2.4.2.** There is  $C > 0$  such that for all  $(\mathbf{v}, \tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \in V_{3d-2d}$  one has

$$\begin{aligned} & \|\mathbf{v}\|_{H^1(\Omega^0; \mathbb{R}^3)}^2 + \|\tilde{\mathbf{v}}\|_{H^1(\omega; \mathbb{R}^3)}^2 + \|\tilde{\mathbf{w}}\|_{H^1(\omega; \mathbb{R}^3)}^2 \\ & \leq C \left( \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega^0; \mathbb{R}^{3 \times 3})}^2 + \|\nabla' \tilde{\mathbf{v}} + \mathbf{A} \tilde{\mathbf{w}}\|_{L^2(\omega; \mathbb{R}^{3 \times 2})}^2 + \|\nabla' \tilde{\mathbf{w}}\|_{L^2(\omega; \mathbb{R}^{3 \times 2})}^2 \right). \end{aligned}$$

*Proof.* Let us suppose the opposite. Then there is a sequence  $(\mathbf{v}^k, \tilde{\mathbf{v}}^k, \tilde{\mathbf{w}}^k) \in V_{3d-2d}$  such that

$$\|\mathbf{v}^k\|_{H^1(\Omega^0; \mathbb{R}^3)}^2 + \|\tilde{\mathbf{v}}^k\|_{H^1(\omega; \mathbb{R}^3)}^2 + \|\tilde{\mathbf{w}}^k\|_{H^1(\omega; \mathbb{R}^3)}^2 = 1 \quad (2.4.1)$$

and

$$\begin{aligned} \mathbf{v}^k &\rightharpoonup \mathbf{v}^0 && \text{weakly in } H^1(\Omega^0; \mathbb{R}^3), \\ \tilde{\mathbf{v}}^k &\rightharpoonup \tilde{\mathbf{v}}^0 && \text{weakly in } H^1(\omega; \mathbb{R}^3), \\ \tilde{\mathbf{w}}^k &\rightharpoonup \tilde{\mathbf{w}}^0 && \text{weakly in } H^1(\omega; \mathbb{R}^3), \end{aligned} \quad (2.4.2)$$

$$\|\mathbf{e}(\mathbf{v}^k)\|_{L^2(\Omega^0; \mathbb{R}^{3 \times 3})}^2 + \|\nabla' \tilde{\mathbf{v}}^k + \mathbf{A} \tilde{\mathbf{w}}^k\|_{L^2(\omega; \mathbb{R}^{3 \times 2})}^2 + \|\nabla' \tilde{\mathbf{w}}^k\|_{L^2(\omega; \mathbb{R}^{3 \times 2})}^2 \rightarrow 0.$$

By the trace theorem  $\tilde{\mathbf{v}}^0 = \mathbf{v}^0|_{x_3=0}$ . The last convergence in (2.4.2) and the Korn inequality implies that the first convergence in (2.4.2) is strong and that  $\mathbf{v}^0 = 0$ . Thus also  $\tilde{\mathbf{v}}^0 = 0$ . From the last convergence we also get that the second and third convergence are also strong in  $H^1(\omega; \mathbb{R}^3)$ . Furthermore we obtain that

$$\nabla' \tilde{\mathbf{v}}^0 + \mathbf{A} \tilde{\mathbf{w}}^0 = 0.$$

Since  $\tilde{\mathbf{v}}^0 = 0$  this implies that  $\mathbf{A} \tilde{\mathbf{w}}^0 = 0$ , which implies  $\tilde{\mathbf{w}}^0 = 0$ . This contradicts (2.4.1) since the norm is preserved for strongly convergent sequences.  $\square$

Using the definition of  $V_{3d-2d}$ , the trace theorem and the classical Korn inequality we obtain

$$\|\tilde{\mathbf{v}}\|_{L^2(\omega; \mathbb{R}^3)} \leq C \|\mathbf{v}|_{x_3=0}\|_{L^2(\omega; \mathbb{R}^3)} \leq C \|\mathbf{v}\|_{H^1(\Omega^0; \mathbb{R}^3)} \leq C \|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega^0; \mathbb{R}^9)}.$$

Therefore, we obtain the following a priori estimates.

**Lemma 2.4.3.** Let  $(\mathbf{u}^h, \tilde{\mathbf{u}}^h, \tilde{\mathbf{w}}^h) \in V_{3d-2d}$  be the solution of the problem (2.2.4). Then for all  $\alpha \geq -1$  there is  $C > 0$  such that for all  $h$  the following estimates hold

$$\|\mathbf{e}(\mathbf{u}^h)\|_{L^2(\Omega^0; \mathbb{R}^9)}, h^{-\alpha/2} \|\nabla' \tilde{\mathbf{u}}^h + \mathbf{A} \tilde{\mathbf{w}}^h\|_{L^2(\Omega^0; \mathbb{R}^{3 \times 2})}, h^{(2-\alpha)/2} \|\nabla' \tilde{\mathbf{w}}^h\|_{L^2(\Omega^0; \mathbb{R}^{3 \times 2})} \leq C.$$

Using the Korn inequality on  $\Omega^0$  from the a priori estimates from Lemma 2.4.3 we obtain the following convergence result.

**Corollary 2.4.4.** There is a sequence  $(h_k)_k$  such that  $h_k \rightarrow 0$  and  $\mathbf{u}^0 \in V(\Omega^0)$  and  $\mathbf{e}^m, \mathbf{e}^f \in L^2(\Omega^0; \mathbb{R}^{3 \times 2})$  such that

$$\begin{aligned} \mathbf{u}^{h_k} &\rightharpoonup \mathbf{u}^0 && \text{weakly in } H^1(\Omega^0; \mathbb{R}^3), \\ h_k^{-\alpha/2} (\nabla' \tilde{\mathbf{u}}^{h_k} + \mathbf{A} \tilde{\mathbf{w}}^{h_k}) &\rightharpoonup \mathbf{e}^m && \text{weakly in } L^2(\omega; \mathbb{R}^{3 \times 2}), \\ h_k^{(2-\alpha)/2} \nabla' \tilde{\mathbf{w}}^{h_k} &\rightharpoonup \mathbf{e}^f && \text{weakly in } L^2(\omega; \mathbb{R}^{3 \times 2}). \end{aligned} \quad (2.4.3)$$

**Theorem 2.4.5.** Let  $\alpha \geq -1$  and let  $(\mathbf{u}^h, \tilde{\mathbf{u}}^h, \tilde{\mathbf{w}}^h) \in V_{3d-2d}$  be the solution of the problem (2.2.4). Then

$$\begin{aligned} \mathbf{u}^h &\rightarrow \mathbf{u}^0 && \text{strongly in } H^1(\Omega^0; \mathbb{R}^3); \\ \tilde{\mathbf{u}}^h &\rightarrow \tilde{\mathbf{u}}^0 && \text{strongly in } L^2(\omega; \mathbb{R}^3); \\ h^{-\alpha/2} (\nabla' \tilde{\mathbf{u}}^h + \mathbf{A} \tilde{\mathbf{w}}^h) &\rightarrow \mathbf{e}^m && \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}), \\ h^{(2-\alpha)/2} \nabla' \tilde{\mathbf{w}}^h &\rightarrow \mathbf{e}^f && \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}). \end{aligned} \quad (2.4.4)$$

The limit function  $\mathbf{u}^0$  is such that  $\tilde{\mathbf{u}}^0 = \mathbf{u}^0|_{x_3=0}$  and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^I$ . Furthermore  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0)$ ,  $\mathbf{e}^m$  and  $\mathbf{e}^f$  are uniquely determined by:

I) for  $-1 \leq \alpha < 0$ ,  $\mathbf{e}^m = \mathbf{e}^f = 0$  and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^I$  is the unique solution of (2.2.5).

II) for  $\alpha = 0$

$$\mathbf{e}^m = \begin{bmatrix} \partial_1 \tilde{u}_1^0 & \frac{1}{2}(\partial_1 \tilde{u}_2^0 + \partial_2 \tilde{u}_1^0) \\ \frac{1}{2}(\partial_1 \tilde{u}_2^0 + \partial_2 \tilde{u}_1^0) & \partial_2 \tilde{u}_2^0 \\ 0 & 0 \end{bmatrix},$$

$\mathbf{e}^f = 0$  and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{II}$  is the unique solution of (2.2.6). Furthermore

$$(\tilde{u}_1^h, \tilde{u}_2^h) \rightarrow (\tilde{u}_1^0, \tilde{u}_2^0) \quad \text{strongly in } H^1(\omega; \mathbb{R}^2).$$

III) for  $0 < \alpha < 2$ ,  $\mathbf{e}^m = \mathbf{e}^f = 0$  and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{III}$  is the unique solution of (2.2.7).

Furthermore

$$\tilde{\omega}^h \rightharpoonup \tilde{\omega}^0 = (\partial_2 \tilde{u}_3^0, -\partial_1 \tilde{u}_3^0, \partial_1 \tilde{u}_2^0) \quad \text{weakly in } H^{-1}(\omega; \mathbb{R}^3). \quad (2.4.5)$$

IV) for  $\alpha = 2$   $\mathbf{e}^m = 0$ ,

$$\mathbf{e}^f = \begin{bmatrix} \partial_{12} \tilde{u}_3^0 & \partial_{22} \tilde{u}_3^0 \\ -\partial_{11} \tilde{u}_3^0 & -\partial_{12} \tilde{u}_3^0 \\ 0 & 0 \end{bmatrix}.$$

and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{IV}$  is the unique solution of (2.2.8). Furthermore, the second convergence in (2.4.4) is strong in  $H^1(\omega; \mathbb{R}^3)$  and the convergence in (2.4.5) is strong in  $H^1(\omega; \mathbb{R}^3)$ .

V) for  $2 < \alpha$ ,  $\mathbf{e}^m = \mathbf{e}^f = 0$  and  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^V$  is the unique solution of (2.2.9). As in the previous case, the second convergence in (2.4.4) is also strong in  $H^1(\omega; \mathbb{R}^3)$  and the convergence in (2.4.5) is strong in  $H^1(\omega; \mathbb{R}^3)$ .

### 2.4.1. 3d–2d model: case $-1 \leq \alpha < 0$

Let  $V(\Omega^0) = H^1(\Omega^0; \mathbb{R}^3)$ . From Corollary 2.4.4 we have that convergences (2.4.3) hold. Thus we can take the limit in the model (2.2.4), when  $k$  tends to infinity, and obtain that the limit function  $\mathbf{u}^0 \in V(\Omega^0)$  satisfies

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \in V_{3d-2d}.$$

Thus  $\mathbf{u}^0 \in V(\Omega^0)$  satisfies

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}', \quad \mathbf{v} \in V(\Omega^0). \quad (2.4.6)$$

This is the problem (2.2.5) obtained already in (2.3.6) and possesses unique solution. Therefore the whole family  $(\mathbf{u}^h)_{h>0}$  converges to  $\mathbf{u}^0$ . Note also that according to the trace theorem then  $\tilde{\mathbf{u}}^h = \mathbf{u}^h$  converges weakly in  $L^2(\omega; \mathbb{R}^3)$ .

Let us define the sequence

$$\begin{aligned} \Lambda(k) &= \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^{h_k} - \mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^{h_k} - \mathbf{u}^0) d\mathbf{x} \\ &+ \int_{\omega} \mathcal{C}_m \left( h_k^{-\alpha/2} (\nabla' \tilde{\mathbf{u}}^{h_k} + \mathbf{A} \tilde{\boldsymbol{\omega}}^{h_k}) - \mathbf{e}^m \right) \cdot \left( h_k^{-\alpha/2} (\nabla' \tilde{\mathbf{u}}^{h_k} + \mathbf{A} \tilde{\boldsymbol{\omega}}^{h_k}) - \mathbf{e}^m \right) d\mathbf{x}' \\ &+ \frac{1}{12} \int_{\omega} \mathcal{C}_f \left( h_k^{(2-\alpha)/2} \nabla' \tilde{\boldsymbol{\omega}}^h - \mathbf{e}^f \right) \cdot \left( h_k^{(2-\alpha)/2} \nabla' \tilde{\boldsymbol{\omega}}^h - \mathbf{e}^f \right) d\mathbf{x}'. \end{aligned} \quad (2.4.7)$$

Eliminating the quadratic terms using the equation (2.2.4) we obtain

$$\begin{aligned} \Lambda(k) &= \int_{\omega} \mathbf{f} \cdot \mathbf{u}^{h_k} d\mathbf{x}' - 2 \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^{h_k}) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} + \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} \\ &- 2 \int_{\omega} \mathcal{C}_m \left( h_k^{-\alpha/2} (\nabla' \tilde{\mathbf{u}}^{h_k} + \mathbf{A} \tilde{\boldsymbol{\omega}}^{h_k}) \right) \cdot \mathbf{e}^m d\mathbf{x}' + \int_{\omega} \mathcal{C}^m \mathbf{e}^m \cdot \mathbf{e}^m d\mathbf{x}' \\ &- \frac{1}{6} \int_{\omega} \mathcal{C}_f \left( h_k^{(2-\alpha)/2} \nabla' \tilde{\boldsymbol{\omega}}^{h_k} \right) \cdot \mathbf{e}^f d\mathbf{x}' + \frac{1}{12} \int_{\omega} \mathcal{C}^f \mathbf{e}^f \cdot \mathbf{e}^f d\mathbf{x}'. \end{aligned} \quad (2.4.8)$$

Next we let  $k$  to infinity using convergences (2.3.2) and in the limit obtain

$$\begin{aligned} \Lambda(k) &\rightarrow \Lambda := \int_{\omega} \mathbf{f} \cdot \mathbf{u}^0 d\mathbf{x}' - \int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} \\ &- \int_{\omega} \mathcal{C}^m \mathbf{e}^m \cdot \mathbf{e}^m d\mathbf{x}' - \frac{1}{12} \int_{\omega} \mathcal{C}^f \mathbf{e}^f \cdot \mathbf{e}^f d\mathbf{x}' \\ &= - \int_{\omega} \mathcal{C}^m \mathbf{e}^m \cdot \mathbf{e}^m d\mathbf{x}' - \frac{1}{12} \int_{\omega} \mathcal{C}^f \mathbf{e}^f \cdot \mathbf{e}^f d\mathbf{x}' \end{aligned}$$

since  $\mathbf{u}^0$  satisfies the equation (2.4.6). Since  $\Lambda \geq 0$  we conclude that  $\mathbf{e}^m = \mathbf{e}^f = 0$  and therefore  $\Lambda = 0$ . Now we also have the uniqueness of the limit for the strains in the plate and thus we have that all convergences in (2.4.3) hold for the whole  $h$  family. Further, since  $\Lambda = 0$  we obtain that

$$\begin{aligned} \mathbf{e}(\mathbf{u}(h)) &\rightarrow \mathbf{e}(\mathbf{u}^0) \quad \text{strongly in } L^2(\Omega^0; \mathbb{R}^{3 \times 3}), \\ h^{-\alpha/2} (\nabla' \tilde{\mathbf{u}}^h + \mathbf{A} \tilde{\boldsymbol{\omega}}^h) &\rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}), \\ h^{(2-\alpha)/2} \nabla' \tilde{\boldsymbol{\omega}}^h &\rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}). \end{aligned}$$

Then the classical Korn inequality implies that the convergences in (2.4.3) are all strong.



2.4.2. 3d–2d model: case  $\alpha = 0$ 

Let  $\sigma^m = \mathcal{C}_m \mathbf{e}^m$ . Now we can take the limit in the model (2.2.4), when  $k$  tends to infinity, and obtain that the limit functions  $\mathbf{u}^0 \in V(\Omega^0)$  and  $\sigma^m \in L^2(\Omega^0; \mathbb{R}^{3 \times 2})$  satisfy

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\omega} \sigma^m \cdot (\nabla' \tilde{\mathbf{v}} + \mathbf{A} \tilde{\mathbf{w}}) d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \in V_{3d-2d}. \quad (2.4.9)$$

Thus for all test functions  $\tilde{\mathbf{w}} \in H^1(\omega; \mathbb{R}^3)$  we obtain

$$\begin{aligned} 0 &= \int_{\omega} \sigma^m \cdot (\mathbf{e}_1 \times \tilde{\mathbf{w}} \mathbf{e}_2 \times \tilde{\mathbf{w}}) d\mathbf{x}' = \int_{\omega} \sigma^m \mathbf{e}_1 \cdot \mathbf{e}_1 \times \tilde{\mathbf{w}} + \sigma^m \mathbf{e}_2 \cdot \mathbf{e}_2 \times \tilde{\mathbf{w}} d\mathbf{x}' \\ &= \int_{\omega} \tilde{\mathbf{w}} \cdot (\sigma^m \mathbf{e}_1 \times \mathbf{e}_1 + \sigma^m \mathbf{e}_2 \times \mathbf{e}_2) d\mathbf{x}'. \end{aligned}$$

Since  $\tilde{\mathbf{w}}$  is arbitrary we obtain

$$\sigma_{31}^m = \sigma_{32}^m = \sigma_{21}^m - \sigma_{12}^m = 0.$$

By the definition of  $\mathcal{C}_m$  this implies  $e_{31}^m = e_{32}^m = e_{21}^m - e_{12}^m = 0$ . Now the second convergence in (2.4.3) implies by components

$$\begin{aligned} \partial_1 \tilde{u}_1^{h_k} &\rightharpoonup e_{11}^m, & \partial_2 \tilde{u}_1^{h_k} + \tilde{\omega}_3^{h_k} &\rightharpoonup e_{12}^m, \\ \partial_1 \tilde{u}_2^{h_k} - \tilde{\omega}_3^{h_k} &\rightharpoonup e_{21}^m, & \partial_2 \tilde{u}_2^{h_k} &\rightharpoonup e_{22}^m, \\ \partial_1 \tilde{u}_3^{h_k} + \tilde{\omega}_2^{h_k} &\rightharpoonup 0, & \partial_2 \tilde{u}_3^{h_k} - \tilde{\omega}_1^{h_k} &\rightharpoonup 0, \end{aligned} \quad (2.4.10)$$

all weakly in  $L^2(\omega)$ . Since  $\tilde{\mathbf{u}}^{h_k} = \mathbf{u}^{h_k}|_{x_3=0}$  from the trace theorem we obtain that

$$\tilde{\mathbf{u}}^{h_k} \rightharpoonup \tilde{\mathbf{u}}^0 \quad \text{weakly in } L^2(\omega; \mathbb{R}^3)$$

and thus  $\tilde{\mathbf{u}}^0 \in L^2(\omega; \mathbb{R}^3)$ , i.e.  $(\mathbf{u}, \tilde{\mathbf{u}}) \in V^I$ . From the (2.4.10) we now identify the limits

$$e_{11}^m = \partial_1 \tilde{u}_1^0, \quad e_{22}^m = \partial_2 \tilde{u}_2^0, \quad e_{12}^m + e_{21}^m = \partial_2 \tilde{u}_1^0 + \partial_2 \tilde{u}_1^0$$

and obtain that  $\tilde{u}_1^0$  and  $\tilde{u}_2^0$  are also in  $H^1(\omega)$ . Therefore  $(\mathbf{u}, \tilde{\mathbf{u}}) \in V^{II}$ . Since  $e_{12}^m = e_{21}^m$  we obtain that also

$$e_{12}^m = e_{21}^m = \frac{1}{2}(\partial_2 \tilde{u}_1^0 + \partial_2 \tilde{u}_1^0).$$

Therefore

$$\mathbf{e}^m = \begin{bmatrix} \partial_1 \tilde{u}_1^0 & \frac{1}{2}(\partial_2 \tilde{u}_1^0 + \partial_2 \tilde{u}_1^0) \\ \frac{1}{2}(\partial_2 \tilde{u}_1^0 + \partial_2 \tilde{u}_1^0) & \partial_2 \tilde{u}_2^0 \\ 0 & 0 \end{bmatrix}.$$

We insert this into (2.4.9) and obtain

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\omega} C_m \mathbf{e}^m \cdot \nabla' \tilde{\mathbf{v}} d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}, \tilde{\mathbf{u}}) \in V_{3d-2d}.$$

Thus, since  $C_m \mathbf{e}^m$  is symmetric and since the projection on the first two components of  $V_{3d-2d}$  is dense in  $V^{II}$ , we obtain that the limit  $(\mathbf{u}, \tilde{\mathbf{u}}) \in V^{II}$  satisfies

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \int_{\omega} \mathcal{A} \mathbf{e}'(\tilde{\mathbf{u}}) \cdot \mathbf{e}'(\tilde{\mathbf{v}}) d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^{II}. \quad (2.4.11)$$

Here  $\mathbf{e}'(\tilde{\mathbf{v}}) = \frac{1}{2}(\nabla'(\tilde{v}_1, \tilde{v}_2) + \nabla'(\tilde{v}_1, \tilde{v}_2)^T)$ . This problem is given in (2.2.6) and has been already obtained in (2.3.11). As shown before its solution is unique. Therefore the whole family  $(\mathbf{u}^h)_{h>0}$  converges to  $\mathbf{u}^0$ . Note also that according to the trace theorem  $\tilde{\mathbf{u}}^h = \mathbf{u}^h$  converges weakly in  $L^2(\omega; \mathbb{R}^3)$  to  $\tilde{\mathbf{u}}$ .

Next we define  $\Lambda(k)$  as in the previous subsection, i.e., as in (2.4.7). Then we use (2.2.4) and eliminate the quadratic terms and then let  $k$  to infinity. As before we obtain

$$\begin{aligned} \Lambda(k) \rightarrow \Lambda &:= \int_{\omega} \mathbf{f} \cdot \mathbf{u}^0 d\mathbf{x}' - \int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} \\ &\quad - \int_{\omega} C^m \mathbf{e}^m \cdot \mathbf{e}^m d\mathbf{x}' - \frac{1}{12} \int_{\omega} C^f \mathbf{e}^f \cdot \mathbf{e}^f d\mathbf{x}'. \end{aligned}$$

Now the application of the model (2.4.11) and knowledge of  $\mathbf{e}^m$  implies

$$\begin{aligned} \Lambda &= \int_{\omega} \mathcal{A} \mathbf{e}'(\tilde{\mathbf{u}}) \cdot \mathbf{e}'(\tilde{\mathbf{v}}) d\mathbf{x}' - \int_{\omega} C^m \mathbf{e}^m \cdot \mathbf{e}^m d\mathbf{x}' - \frac{1}{12} \int_{\omega} C^f \mathbf{e}^f \cdot \mathbf{e}^f d\mathbf{x}' \\ &= -\frac{1}{12} \int_{\omega} C^f \mathbf{e}^f \cdot \mathbf{e}^f d\mathbf{x}', \end{aligned}$$

since  $C^m \mathbf{e}^m \cdot \mathbf{e}^m = \mathcal{A} \mathbf{e}'(\tilde{\mathbf{u}}) \cdot \mathbf{e}'(\tilde{\mathbf{u}})$ . Therefore  $\mathbf{e}^f = 0$  and

$$\begin{aligned} \mathbf{e}(\mathbf{u}^h) &\rightarrow \mathbf{e}(\mathbf{u}^0) \quad \text{strongly in } L^2(\Omega^0; \mathbb{R}^{3 \times 3}), \\ (\nabla' \tilde{\mathbf{u}}^h + \mathbf{A} \tilde{\boldsymbol{\omega}}^h) &\rightarrow \mathbf{e}^m \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}), \\ h \nabla' \tilde{\boldsymbol{\omega}}^h &\rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}). \end{aligned}$$

From the second convergence we have that convergences in (2.4.10) are strong. Thus we have  $\mathbf{e}'(\tilde{\mathbf{u}}^h) \rightarrow \mathbf{e}'(\tilde{\mathbf{u}})$  strongly in  $L^2(\omega; \mathbb{R}^{2 \times 2})$ . The Korn inequality on  $H^1(\omega; \mathbb{R}^2)$  gives

$$\|(\tilde{u}_1^h, \tilde{u}_2^h)\|_{H^1(\omega; \mathbb{R}^2)}^2 \leq C(\|\mathbf{e}'(\tilde{\mathbf{u}}^h)\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + \|\tilde{\mathbf{u}}^h\|_{L^2(\omega; \mathbb{R}^2)}^2). \quad (2.4.12)$$

By the continuity of the trace operator from  $H^1(\Omega^0; \mathbb{R}^3)$  to  $L^2(\omega; \mathbb{R}^3)$  we have that  $\tilde{\mathbf{u}}^h \rightarrow \tilde{\mathbf{u}}^0$  strongly in  $L^2(\omega; \mathbb{R}^3)$ . Now the Korn inequality (2.4.12) implies that  $\mathbf{u}^h \rightarrow \mathbf{u}^0$  strongly in  $H^1(\omega^0; \mathbb{R}^3)$  and  $(\tilde{u}_1^h, \tilde{u}_2^h)$  strongly to  $(\tilde{u}_1^0, \tilde{u}_2^0)$  in  $H^1(\omega; \mathbb{R}^2)$ .

2.4.3. 3d–2d model: case  $0 < \alpha < 2$ 

From the first convergence in (2.4.3) and the trace theorem we obtain

$$\tilde{\mathbf{u}}^{h_k} \rightharpoonup \tilde{\mathbf{u}}^0 \quad \text{weakly in } L^2(\omega; \mathbb{R}^3). \quad (2.4.13)$$

From the second convergence, since  $-\alpha > 0$ , we obtain that

$$(\partial_1 \tilde{\mathbf{u}}^{h_k} + \mathbf{e}_1 \times \tilde{\boldsymbol{\omega}}^{h_k} \partial_2 \tilde{\mathbf{u}}^{h_k} + \mathbf{e}_2 \times \tilde{\boldsymbol{\omega}}^{h_k}) \rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}).$$

This implies that

$$\begin{aligned} \mathbf{e}'(\tilde{\mathbf{u}}^{h_k}) &\rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{2 \times 2}), \text{ and} \\ \partial_1 \tilde{u}_2^{h_k} - \tilde{\omega}_3^{h_k} &\rightarrow 0, \quad \partial_2 \tilde{u}_1^{h_k} + \tilde{\omega}_3^{h_k} \rightarrow 0, \\ \partial_1 \tilde{u}_3^{h_k} + \tilde{\omega}_2^{h_k} &\rightarrow 0, \quad \partial_2 \tilde{u}_3^{h_k} - \tilde{\omega}_1^{h_k} \rightarrow 0 \quad \text{strongly in } L^2(\omega). \end{aligned} \quad (2.4.14)$$

From (2.4.13), by the uniqueness of the limit we obtain that  $\mathbf{e}'(\tilde{\mathbf{u}}) = 0$  and further

$$\tilde{\omega}_3^{h_k} \rightharpoonup \partial_1 \tilde{u}_2^0 = -\partial_2 \tilde{u}_1^0, \quad \tilde{\omega}_2^{h_k} \rightharpoonup -\partial_1 \tilde{u}_3^0, \quad \tilde{\omega}_1^{h_k} \rightharpoonup \partial_2 \tilde{u}_3^0 \quad \text{weakly in } H^{-1}(\omega). \quad (2.4.15)$$

This implies that the limit longitudinal displacement of the plate is an infinitesimal rigid displacement, i.e.,

$$\begin{aligned} (\tilde{\mathbf{u}}^0, \tilde{\boldsymbol{\omega}}^0) &\in \{(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\omega}}) \in H^1(\omega; \mathbb{R}^3) \times L^2(\omega; \mathbb{R}^3) \\ &\quad : \partial_1 \tilde{\mathbf{v}} + \mathbf{e}_1 \times \tilde{\boldsymbol{\omega}} = \partial_2 \tilde{\mathbf{v}} + \mathbf{e}_2 \times \tilde{\boldsymbol{\omega}} = 0\} =: W^{III}. \end{aligned}$$

The two conditions in this space imply that  $\mathbf{e}'(\tilde{\mathbf{v}}) = 0$  and that the infinitesimal rotation  $\tilde{\boldsymbol{\omega}}$  can be expressed in terms of derivatives of displacement

$$\tilde{w}_1 = \partial_2 \tilde{v}_3, \quad \tilde{w}_2 = -\partial_1 \tilde{v}_3, \quad \tilde{w}_3 = \partial_1 \tilde{v}_2.$$

Therefore the limit of  $(\mathbf{u}^{h_k}, \tilde{\mathbf{u}}^{h_k})$ ,  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0)$  belongs to the space  $V^{III}$ .

After taking the limit in (2.2.4) for the test function  $(\mathbf{v}, \tilde{\mathbf{v}}, \tilde{\boldsymbol{\omega}}) \in W^{III}$  the limit  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{III}$  satisfies

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^{III}. \quad (2.4.16)$$

This is problem (2.2.7), already obtained in (2.3.16). Its solution is unique. Therefore the whole family  $(\mathbf{u}^h)_{h>0}$  converges to  $\mathbf{u}^0$ . Note also that according to the trace theorem

$\tilde{\mathbf{u}}^h = \mathbf{u}^h$  converges weakly in  $L^2(\omega; \mathbb{R}^3)$  to  $\tilde{\mathbf{u}}$ . Additionally,  $\mathbf{e}'(\tilde{\mathbf{u}}^h)$  converges to zero strongly in  $L^2(\omega; \mathbb{R}^{2 \times 2})$ . Thus, by the Korn inequality we obtain the strong convergence of  $(\tilde{u}_1^h, \tilde{u}_2^h)$  in  $H^1(\omega; \mathbb{R}^2)$ .

In this case we can repeat the arguments from the end of Subsection 2.4.1 and prove that  $\mathbf{e}^m = \mathbf{e}^f = 0$  and that the whole families are convergent and that all convergences in (2.4.3) are in fact strong. This holds since the model equation is the same for both limit models and sets the first line in definition of  $\Lambda$  in (2.4.8) to zero.

#### 2.4.4. 3d–2d model: case $\alpha = 2$

From the first convergence in (2.4.3) and the trace theorem we obtain

$$\tilde{\mathbf{u}}^{h_k} \rightharpoonup \tilde{\mathbf{u}}^0 \quad \text{weakly in } L^2(\omega; \mathbb{R}^3). \quad (2.4.17)$$

From the second convergence we obtain that

$$(\partial_1 \tilde{\mathbf{u}}^{h_k} + \mathbf{e}_1 \times \tilde{\boldsymbol{\omega}}^{h_k} \partial_2 \tilde{\mathbf{u}}^{h_k} + \mathbf{e}_2 \times \tilde{\boldsymbol{\omega}}^{h_k}) \rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}).$$

This implies that

$$\begin{aligned} \mathbf{e}'(\tilde{\mathbf{u}}^{h_k}) &\rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{2 \times 2}), \text{ and} \\ \partial_1 \tilde{u}_2^{h_k} - \tilde{\omega}_3^{h_k} &\rightarrow 0, \quad \partial_2 \tilde{u}_1^{h_k} + \tilde{\omega}_3^{h_k} \rightarrow 0, \\ \partial_1 \tilde{u}_3^{h_k} + \tilde{\omega}_2^{h_k} &\rightarrow 0, \quad \partial_2 \tilde{u}_3^{h_k} - \tilde{\omega}_1^{h_k} \rightarrow 0 \quad \text{strongly in } L^2(\omega). \end{aligned} \quad (2.4.18)$$

From (2.4.17), by the uniqueness of the limit we obtain that  $\mathbf{e}'(\tilde{\mathbf{u}}) = 0$  and further

$$\tilde{\omega}_3^{h_k} \rightharpoonup \partial_1 \tilde{u}_2^0 = -\partial_2 \tilde{u}_1^0, \quad \tilde{\omega}_2^{h_k} \rightharpoonup -\partial_1 \tilde{u}_3^0, \quad \tilde{\omega}_1^{h_k} \rightharpoonup \partial_2 \tilde{u}_3^0 \quad \text{weakly in } H^{-1}(\omega). \quad (2.4.19)$$

Together with the third convergence in (2.4.3) and the Lions lemma ( $f \in L^2$  is equivalent to  $f \in H^{-1}$  and  $\nabla' f \in H^{-1}$ ) this implies that the convergences in (2.4.19) are in fact in  $L^2(\omega)$  (and the limit functions are in  $L^2(\omega)$  as well). Again, together with the third convergence in (2.4.3) this implies that the convergences in (2.4.19) are actually weak in  $H^1(\omega)$  (and the limit functions are in  $H^1(\omega)$  as well and thus  $\tilde{\mathbf{u}}^0 \in H^2(\omega; \mathbb{R}^3)$ ). Then from (2.4.18) using the Korn inequality this implies that

$$\begin{aligned} \tilde{\mathbf{u}}^{h_k} &\rightarrow \tilde{\mathbf{u}}^0 \quad \text{strongly in } H^1(\omega; \mathbb{R}^3), \\ \tilde{\boldsymbol{\omega}}^{h_k} &\rightarrow \tilde{\boldsymbol{\omega}}^0 = (\partial_2 \tilde{u}_3^0, -\partial_1 \tilde{u}_3^0, \partial_1 \tilde{u}_2^0)^T \quad \text{weakly in } H^1(\omega; \mathbb{R}^3), \\ \mathbf{e}'(\mathbf{u}^0) &= 0. \end{aligned} \quad (2.4.20)$$

Further from the third convergence in (2.4.3) we obtain that

$$\mathbf{e}^f = \nabla' \tilde{\omega}^0 = \begin{bmatrix} \partial_{12} \tilde{u}_3^0 & \partial_{22} \tilde{u}_3^0 \\ -\partial_{11} \tilde{u}_3^0 & -\partial_{12} \tilde{u}_3^0 \\ -\partial_{12} \tilde{u}_2^0 & -\partial_{12} \tilde{u}_2^0 \end{bmatrix} = \begin{bmatrix} \partial_{12} \tilde{u}_3^0 & \partial_{22} \tilde{u}_3^0 \\ -\partial_{11} \tilde{u}_3^0 & -\partial_{12} \tilde{u}_3^0 \\ 0 & 0 \end{bmatrix}.$$

Since  $\mathbf{e}^f \in L^2(\omega; \mathbb{R}^{3 \times 2})$  we obtain that  $D^2 \tilde{u}_3^0 \in L^2(\omega; \mathbb{R}^{2 \times 2})$  and thus  $\tilde{u}_3^0 \in H^2(\omega)$ . Therefore the limit  $(\mathbf{u}, \tilde{\mathbf{u}})$  belongs to  $V^{IV}$  and  $\tilde{\omega}$  is expressed using derivatives of  $\tilde{u}_3^0$ .

Let us now take the test function  $(\mathbf{v}, \tilde{\mathbf{v}}, \tilde{\omega})$  that satisfies the same conditions as the limit, i.e.,

$$\partial_1 \tilde{\mathbf{v}} + \mathbf{e}_1 \times \tilde{\omega} = \partial_2 \tilde{\mathbf{v}} + \mathbf{e}_2 \times \tilde{\omega} = 0 \quad (2.4.21)$$

and for such test function take the limit in (2.2.4). We obtain that the limit  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0, \tilde{\omega}^0)$  satisfies

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \frac{1}{12} \int_{\omega} C_f \mathbf{e}^f \cdot \nabla' \tilde{\omega} d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}'.$$

Inserting  $\mathbf{e}^f$  and expressing  $\tilde{\omega}$  in terms of  $\tilde{\mathbf{v}}$  from (2.4.21) we obtain that the limit belongs to  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0) \in V^{IV}$  and satisfies

$$\int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} + \frac{1}{12} \int_{\omega} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{v}_3 d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^{IV}. \quad (2.4.22)$$

This problem is the same as the one in (2.2.8) and (2.3.25). Its solution is unique. Therefore the whole families  $(\tilde{\mathbf{u}}^h, \tilde{\omega}^h)_h$  in (2.4.20) converge.

Next we define  $\Lambda(k)$  as in Subsection 2.4.1, i.e., as in (2.4.7). Then we use (2.2.4) and eliminate the quadratic terms and then let  $k$  to infinity. As before we obtain that  $\Lambda(k)$  converges to

$$\Lambda := \int_{\omega} \mathbf{f} \cdot \mathbf{u}^0 d\mathbf{x}' - \int_{\Omega^0} C^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{u}^0) d\mathbf{x} - \int_{\omega} C^m \mathbf{e}^m \cdot \mathbf{e}^m d\mathbf{x}' - \frac{1}{12} \int_{\omega} C^f \mathbf{e}^f \cdot \mathbf{e}^f d\mathbf{x}'.$$

Now the application of the model (2.4.22) and knowledge of  $\mathbf{e}^f$  implies

$$\begin{aligned} \Lambda &= \frac{1}{12} \int_{\omega} \mathcal{A} D^2 \tilde{u}_3^0 \cdot D^2 \tilde{u}_3^0 d\mathbf{x}' - \int_{\omega} C^m \mathbf{e}^m \cdot \mathbf{e}^m d\mathbf{x}' - \frac{1}{12} \int_{\omega} C^f \mathbf{e}^f \cdot \mathbf{e}^f d\mathbf{x}' \\ &= -\frac{1}{12} \int_{\omega} C^m \mathbf{e}^m \cdot \mathbf{e}^m d\mathbf{x}', \end{aligned}$$

since (by  $\mathbf{P} \in \mathbb{R}^{2 \times 3}$  we denote the projector defined by  $P_{ij} = \delta_{ij}$ )

$$C^f \mathbf{e}^f \cdot \mathbf{e}^f = \mathcal{A} (\mathbf{J} \mathbf{P} \mathbf{e}^f) \cdot \mathbf{J} \mathbf{P} \mathbf{e}^f = \mathcal{A} (D^2 \tilde{u}_3^0) \cdot D^2 \tilde{u}_3^0.$$

Therefore  $\mathbf{e}^m = 0$  and

$$\begin{aligned}\mathbf{e}(\mathbf{u}^h) &\rightarrow \mathbf{e}(\mathbf{u}^0) \quad \text{strongly in } L^2(\Omega^0; \mathbb{R}^{3 \times 3}), \\ \frac{1}{h} (\nabla' \tilde{\mathbf{u}}^h + \mathbf{A} \tilde{\boldsymbol{\omega}}^h) &\rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}), \\ \nabla' \tilde{\boldsymbol{\omega}}^h &\rightarrow \nabla' \tilde{\boldsymbol{\omega}}^0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}).\end{aligned}$$

Now by the Korn inequality we obtain that  $\mathbf{u}^h \rightarrow \mathbf{u}^0$  strongly in  $H^1(\Omega^0; \mathbb{R}^3)$ . From the second convergence and the second in (2.4.20) we have that

$$\tilde{\mathbf{u}}^h \rightarrow \tilde{\mathbf{u}}^0 \quad \text{strongly in } H^1(\omega; \mathbb{R}^3).$$

Uniqueness of all limits implies that the whole  $h$  families are convergent. Further, from (2.4.20) and the strong convergence of  $\nabla' \tilde{\boldsymbol{\omega}}^h$  in  $L^2$  we have the strong convergence of  $\tilde{\boldsymbol{\omega}}^h$  in  $H^1(\omega; \mathbb{R}^3)$ .

#### 2.4.5. 3d–2d model: case $\alpha > 2$

From the second and the third convergence in (2.4.3) we obtain that

$$\begin{aligned}(\partial_1 \tilde{\mathbf{u}}^{h_k} + \mathbf{e}_1 \times \tilde{\boldsymbol{\omega}}^{h_k} \partial_2 \tilde{\mathbf{u}}^{h_k} + \mathbf{e}_2 \times \tilde{\boldsymbol{\omega}}^{h_k}) &\rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}), \\ \nabla' \tilde{\boldsymbol{\omega}}^{h_k} &\rightarrow 0 \quad \text{strongly in } L^2(\omega; \mathbb{R}^{3 \times 2}).\end{aligned}$$

As in the case  $\alpha = 2$  this implies that (2.4.20) holds. Furthermore we obtain that  $\nabla' \tilde{\boldsymbol{\omega}}^0 = 0$ , which implies that  $D^2 \tilde{u}_3^0 = 0$ . Therefore the limit  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0)$  belongs to  $V^V$ .

Let us now take the test function  $(\mathbf{v}, \tilde{\mathbf{v}}, \tilde{\mathbf{w}})$  that satisfies the same conditions as the limit

$$(\partial_1 \tilde{\mathbf{v}} + \mathbf{e}_1 \times \tilde{\mathbf{w}} \partial_2 \tilde{\mathbf{v}} + \mathbf{e}_2 \times \tilde{\mathbf{w}}) = \nabla' \tilde{\mathbf{w}} = 0$$

and for such test function take the limit in (2.2.4). Thus the limit  $(\mathbf{u}^0, \tilde{\mathbf{u}}^0)$  belongs to  $V^V$  and satisfies

$$\int_{\Omega^0} \mathcal{C}^0 \mathbf{e}(\mathbf{u}^0) \cdot \mathbf{e}(\mathbf{v}) d\mathbf{x} = \int_{\omega} \mathbf{f} \cdot \tilde{\mathbf{v}} d\mathbf{x}', \quad (\mathbf{v}, \tilde{\mathbf{v}}) \in V^V. \quad (2.4.23)$$

This problem is the same as the one in (2.2.9) and (2.3.28). Its solution is unique. Therefore the whole families  $(\tilde{\mathbf{u}}^h, \tilde{\boldsymbol{\omega}}^h)_h$  in (2.4.20) converge.

In this case we can also repeat the arguments from the end of Subsection 2.4.1 and prove that  $\mathbf{e}^m = \mathbf{e}^f = 0$  and that the whole families are convergent and that all convergences in (2.4.3) are in fact strong. This holds from the same reason as in Subsection 2.4.3, since the model equation is the same for both models and sets the first line in definition of  $\Lambda$  to zero.

## **Part II**

### **Nonlinear models of elastic bodies interaction**



# 3. ASYMPTOTICALLY OBTAINED NONLINEAR PLATE MODELS

In the Part II of the thesis we are going to make a similar analysis as in the Part I but in the context of nonlinear modelling. We start with this chapter in which we give an overview of existing results in the literature regarding the asymptotic analysis of nonlinear elasticity models for plate-like bodies, i.e. thin in one direction with plane geometry. For similar overviews of such results, see [39], and [60] for case of the curved geometry.

## 3.1. PROBLEM SETUP

Let  $\omega \subset \mathbb{R}^2$  be a bounded Lipschitz domain and  $h > 0$  a small parameter. Let us define a domain  $\Omega^h := \omega \times \langle 0, h \rangle$  in which the thin body is situated. Let its stored energy function is given by a function  $W^1 : \mathbb{R} \times \mathbb{M}^{3 \times 3} \rightarrow \mathbb{R}$  and let us apply a force to it at  $\Gamma^h := \omega \times \{h\}$ . Then the total energy is given by

$$\int_{\Omega^h} W^1(h; \nabla \bar{\phi}) d\mathbf{x} - \int_{\Gamma^h} \mathbf{f} \cdot \bar{\phi} d\mathbf{x}',$$

where the  $\bar{\phi} : \Omega^h \rightarrow \mathbb{R}$  is a function describing the position  $\bar{\phi}(\mathbf{x})$  of the point  $\mathbf{x} \in \Omega^h$  after deformation of the body. We are interested in finding  $\bar{\phi}$  (possibly satisfying some boundary conditions) for which the total energy is minimal (if there exists such  $\bar{\phi}$ ).

To derive the plate model, not only that we want to find a minimizer  $\bar{\phi}$  for particular domain  $\Omega^h$  (i.e. a particular value  $h > 0$ ), we want to explore the behaviour of those minimizers for different values of  $h > 0$ , and observe their limit when  $h \rightarrow 0$  in some topology. In nonlinear modelling this is usually done by performing  $\Gamma$ -limit. In order to do that we have to be more specific about the definitions.

Firstly, it is convenient and usual in such problems to rescale all problems for different parameters  $h > 0$  to a fixed domain  $\Omega^1$  by using the mapping

$$(\mathbf{x}', x_3) \in \Omega^1 \xrightarrow{r_h} (\mathbf{x}', hx_3) \in \Omega^h.$$

Secondly, it is easy to see that if  $W^1$  is independent of the first variable ( $h$ ), the term  $\int_{\Omega^h} W^1(h; \nabla \bar{\phi}) d\mathbf{x}$  goes to zero as  $h \rightarrow 0$ . Hence, it is natural to scale Lamé coefficients for the thin body so that the body becomes more stiff as the thickness goes to zero. Similarly as in the linear case in Section 2.2, we suppose that the elasticity coefficients (on which the function  $W^1$  depends in an implicit way) for the body of the thickness  $h$  depend on the parameter  $h$  in such way that there is a function  $W_\alpha^1 : \mathbb{M}^{3 \times 3} \rightarrow \mathbb{R}$ , independent of  $h$ , such that

$$W^1(h; \mathbf{X}) = \frac{1}{h^{\alpha+1}} W_\alpha^1(\mathbf{X}) \quad (3.1.1)$$

for all  $h > 0$ ,  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ , and for a real parameter  $\alpha > -1$ . The assumptions on the stored energy function  $W_\alpha^1$  will be different for different values of  $\alpha$  which explains the subscript  $\alpha$  in the energy density function. Thus we obtain that the total energy for a body  $\Omega^h$  on the rescaled domain is described by

$$\frac{1}{h^\alpha} \int_{\Omega^1} W_\alpha^1(\nabla_h \phi) d\mathbf{x} - \int_{\Gamma^1} \mathbf{f} \cdot \phi d\mathbf{x}'$$

where

$$\nabla' \phi := \begin{bmatrix} \partial_1 \phi & \partial_2 \phi \end{bmatrix}, \quad \nabla_h \phi := \begin{bmatrix} \nabla' \phi & \frac{1}{h} \partial_3 \phi \end{bmatrix}.$$

and  $\phi : \Omega^1 \rightarrow \mathbb{R}$ ,  $\phi = \bar{\phi} \circ \mathbf{r}_h$ , is a parametrization of the deformed body defined on the rescaled fixed domain.

The introduction of the function  $W_\alpha^1$  in (3.1.1) in various papers in the literature is often explained as the scalling of the energy functional. Even though it is a fair argument, since in the Chapter 4 we are going to couple two elastic bodies and their energy functionals, for our case it is more convient to introduce  $W_\alpha^1$  as we did through scalling the elasticity coefficients in the thin body.

To make this applicable to the theory of  $\Gamma$ -convergence, let us fix  $q \in \langle 1, +\infty \rangle$  and define a functional  $J_h^{3d, \alpha} : L^q(\Omega^1; \mathbb{R}^3) \rightarrow \bar{\mathbb{R}}$ ,

$$J_h^{3d, \alpha}(\phi) = \begin{cases} \frac{1}{h^\alpha} \int_{\Omega^1} W_\alpha^1(\nabla_h \phi) d\mathbf{x} - \int_{\Gamma^1} \mathbf{f} \cdot \phi d\mathbf{x} & \phi \in \mathcal{B}^{3d, \alpha}, \\ +\infty & \text{otherwise,} \end{cases} \quad (3.1.2)$$

where  $\mathcal{B}^{3d,\alpha} = W^{1,q}(\Omega^1; \mathbb{R}^3)$  is the set of admissible functions and  $\mathbf{f} \in L^{q'}(\Gamma^1; \mathbb{R}^3)$ ,  $q' = (1 - 1/q)^{-1}$ . Note that no Dirichlet boundary conditions are assumed here. For a fixed  $\alpha > -1$  we are interested in the  $\Gamma$ -limit of the family of functionals  $(J_h^{3d,\alpha})_{h>0}$ .

As announced, the assumptions on  $W_\alpha^1$  will be different for different values of  $\alpha > -1$ . The common assumptions and the assumptions we are going to need for the further analysis are:

W1.1)  $W_\alpha^1 : \mathbb{M}^{3 \times 3} \rightarrow [0, +\infty]$  is continuous,

W1.2)  $W_\alpha^1$  is frame indifferent:  $W_\alpha^1(\mathbf{R}\mathbf{X}) = W_\alpha^1(\mathbf{X})$  for  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ ,  $\mathbf{R} \in \text{SO}(3)$ ,

W1.3)  $W_\alpha^1$  satisfies the coercivity condition:  $W_\alpha^1(\mathbf{X}) \geq c_1 \|\mathbf{X}\|_F^q - c_2$ , for  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ , for real constants  $c_1 > 0$ ,  $c_2$ .

The other assumptions we are going to assume for particular  $\alpha > -1$  are:

W2.1) there exists  $C > 0$  such that  $W_\alpha^1(\mathbf{X}) \leq C(1 + \|\mathbf{X}\|_F^q)$ ,

W2.2)  $W_\alpha^1(\mathbf{X}) = +\infty$  if  $\det(\mathbf{X}) \leq 0$ ,

W2.3)  $\forall \delta > 0 \exists C_\delta > 0$  s.t.  $\forall \mathbf{X} \in \mathbb{M}^{3 \times 3}$  with  $\det(\mathbf{X}) \geq \delta$  it holds  $W_\alpha^1(\mathbf{X}) \leq C_\delta(1 + \|\mathbf{X}\|_F^q)$ ,

W2.4)  $W_\alpha^1(\mathbf{X}) \geq c \text{dist}^2(\mathbf{X}, \text{SO}(3))$ , for all  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ , for some  $c > 0$ ,  $W_\alpha^1(\mathbf{R}) = 0$  for  $\mathbf{R} \in \text{SO}(3)$ ,

W2.5)  $W_\alpha^1(\mathbf{X}) \leq C \text{dist}^2(\mathbf{X}, \text{SO}(3))$  in a neighbourhood of  $\text{SO}(3)$ , for some  $C > 0$ ,

W2.6)  $W_\alpha^1 \in C^2$  in the neighbourhood of  $\text{SO}(3)$ .

Clearly, the assumption W2.3) is an adjusted type of growth condition W2.1), and it is going to be used together with W2.2). The assumption W2.4) implies the already assumed coercivity condition W1.3) for  $q = 2$ :

$$W_\alpha^1(\mathbf{X}) \geq c \text{dist}^2(\mathbf{X}, \text{SO}(3)) = c \|\mathbf{X} - \mathbf{R}_\mathbf{X}\|_F^2 \geq \frac{c}{2} \|\mathbf{X}\|_F^2 - c \|\mathbf{R}_\mathbf{X}\|_F^2 = \frac{c}{2} \|\mathbf{X}\|_F^2 - 9c, \quad (3.1.3)$$

where  $\mathbf{R}_\mathbf{X}$  is the  $\text{SO}(3)$  matrix which is the closest to  $\mathbf{X}$ . Similarly the assumption W2.5) implies W2.1) for  $q = 2$ .

**Remark 3.1.1.** Note that for  $W_\alpha^1$  we didn't assume quasiconvexity, the most usual assumption to assure that the functional  $J_h^{3d,\alpha}$  (for particular values of  $\alpha$  and  $h$ ) is weakly lower semicontinuous. Thus, we did not make assumptions to assure that the problem

$$\text{find } \phi^0 \in \mathcal{B}_\alpha^{3d,\alpha}, \quad J_h^{3d,\alpha}(\phi^0) = \min_{\phi \in \mathcal{B}^{3d,\alpha}} J_h^{3d,\alpha}(\phi) \quad (3.1.4)$$

has a solution. This is often in the asymptotic analysis of  $\Gamma$ -limits of functionals describing thin 3d elastic bodies. One can show that (if some Dirichlet boundary conditions are prescribed) each of the functionals  $J_h^{3d,\alpha}$  is uniformly bounded from below, so it attains the infimum. Thus there exists an infimizing sequence  $(\phi_n)_{n \geq 1}$  attaining the infimum in  $\mathcal{B}^{3d,\alpha}$ . As usual this is not a problem since we will analyze the behaviour of the system as  $h \rightarrow 0$  through the  $\Gamma$ -limit of the functionals  $(J_h^{3d,\alpha})_{h>0}$ . Firstly, analogous minimization problem defined for the  $\Gamma$ -limit will have a solution (as a property of  $\Gamma$ -convergence). Secondly, again by properties of  $\Gamma$ -convergence, it will hold that "almost infimizers" of the problem (functions  $\phi^h$  for which the energy  $J_h^{3d,\alpha}$  is  $o(h)$  apart from the infimum) converge (up to a subsequence) to the minimizer of the  $\Gamma$ -limit of the family of functionals  $(J_h^{3d,\alpha})_{h>0}$ .

**Remark 3.1.2.** The  $\Gamma$ -limit of the family of functionals  $(J_h^{3d,\alpha}(\phi))_{h>0}$  will be observed in the strong  $L^q(\Omega^1; \mathbb{R}^3)$  topology. However, due to uniform coercivity properties of the family of functionals (in particular Lemma 3.3.1), the  $\Gamma$ -convergence in the strong  $L^q(\Omega^1; \mathbb{R}^3)$  topology is equivalent to the  $\Gamma$ -convergence in the weak  $W^{1,q}(\Omega^1; \mathbb{R}^3)$  topology.

### 3.2. THE MAIN RESULT

In this section we are going to determine the  $\Gamma$ -limit of families  $(J_h^{3d,\alpha})_{h>0}$  for different values of  $\alpha > -1$ .

In asymptotic analysis using  $\Gamma$ -convergence of functionals describing thin 3d elastic bodies it is natural that in the limit model all admissible functions  $\phi$  do not depend on the third variable. For that reason, limit equations and functions can be defined on two-dimensional domain instead of three-dimensional domain. In other words, we will often use isomorphisms

$$\{\phi \in L^q(\Omega^1; \mathbb{R}^3) : \partial_3 \phi = 0\} \equiv L^q(\omega; \mathbb{R}^3), \quad (3.2.1)$$

and

$$\{\phi \in W^{1,q}(\Omega^1; \mathbb{R}^3) : \partial_3 \phi = 0\} \equiv W^{1,q}(\omega; \mathbb{R}^3). \quad (3.2.2)$$

For that reason we introduce notation using *tilde*: for function  $\phi \in L^q(\Omega^1; \mathbb{R}^3)$  with  $\partial_3 \phi = 0$ , the function  $\tilde{\phi} \in L^q(\omega; \mathbb{R}^3)$  denotes its corresponding function. Also, in the same manner (in the case when  $\phi$  is independent of  $x_3$ ) we will identify the forcing function  $\mathbf{f} \in L^{q'}(\Gamma^1; \mathbb{R}^3)$  with a function from  $L^{q'}(\omega; \mathbb{R}^3)$ . In other words, we will write the forcing term as

$$-\int_{\Gamma^1} \mathbf{f} \cdot \phi d\mathbf{x}' \equiv -\int_{\omega} \mathbf{f} \cdot \phi d\mathbf{x}'.$$

We can now state the main theorem:

**Theorem 3.2.1.** Let the family of functionals  $(J_h^{3d,\alpha})_{h>0}$  be as defined in (3.1.2), with  $W_{\alpha}^1$  satisfying assumptions W1.1), W1.2), W1.3).

- a) Let  $\alpha \in \langle -1, 0 \rangle$ . Let additionally  $W_{\langle -1, 0 \rangle}^1$  satisfies the condition W2.1). Then the  $\Gamma$ -limit of the family of functionals  $(J_h^{3d,\langle -1, 0 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$J_0^{3d,\langle -1, 0 \rangle}(\tilde{\phi}) = \begin{cases} -\int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & \tilde{\phi} \in L^q(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.2.3)$$

- b) Let  $\alpha = 0$ . Let additionally  $W_0^1$  satisfies either the condition [W2.1](#)), or conditions [W2.2](#)) and [W2.3](#)). Then the  $\Gamma$ -limit of the family of functionals  $(J_h^{3d,0})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$J_0^{3d,0}(\tilde{\phi}) = \begin{cases} \int_{\omega} Q\tilde{W}_0^1(\nabla' \tilde{\phi}) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & \tilde{\phi} \in W^{1,q}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \quad (3.2.4)$$

where

$$\tilde{W}_0^1(\mathbf{X}) := \inf_{\mathbf{z} \in \mathbb{R}^3} W_0^1 \left( \begin{bmatrix} \mathbf{X} & \mathbf{z} \end{bmatrix} \right) \quad (3.2.5)$$

for all  $\mathbf{X} \in \mathbb{M}^{3 \times 2}$ .

- c) Let  $\alpha \in \langle 0, 5/3 \rangle$ , and let  $q = 2$ . Let additionally  $W_{\langle 0, 5/3 \rangle}^1$  satisfies conditions [W2.4](#)) and [W2.5](#)). Then the  $\Gamma$ -limit of the family of functionals  $(J_h^{3d, \langle 0, 5/3 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$J_0^{3d, \langle 0, 5/3 \rangle}(\tilde{\phi}) = \begin{cases} - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & \tilde{\phi} \in \mathcal{B}_0^{3d, \langle 0, 5/3 \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (3.2.6)$$

where

$$\mathcal{B}_0^{3d, \langle 0, 5/3 \rangle} := \{ \tilde{\phi} \in W^{1,\infty}(\omega; \mathbb{R}^3) : \nabla' \tilde{\phi}^T \nabla' \tilde{\phi} \leq \mathbf{I} \text{ a.e.} \}. \quad (3.2.7)$$

- d) Let  $\alpha = 2$ , and let  $q = 2$ . Let additionally  $W_2^1$  satisfies conditions [W2.4](#)) and [W2.6](#)). Then the  $\Gamma$ -limit of the family of functionals  $(J_h^{3d,2})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$J_0^{3d,2}(\tilde{\phi}) = \begin{cases} \frac{1}{24} \int_{\omega} Q_2(\mathbf{b}(\tilde{\phi})) d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & \tilde{\phi} \in \mathcal{B}_0^{3d,2}, \\ +\infty & \text{otherwise,} \end{cases} \quad (3.2.8)$$

where

$$\mathcal{B}_0^{3d,2} := \{ \tilde{\phi} \in W^{2,2}(\omega; \mathbb{R}^3) : \nabla \tilde{\phi}^T \nabla \tilde{\phi} = \mathbf{I} \} \quad (3.2.9)$$

and  $\mathbf{b}(\tilde{\phi}) = (b_{\beta_1, \beta_2}(\tilde{\phi}))_{\beta_1, \beta_2}$  is the curvature tensor:

$$b_{\beta_1, \beta_2}(\tilde{\phi}) = \partial_{\beta_1} \mathbf{a}_3^{\tilde{\phi}} \cdot \partial_{\beta_2} \tilde{\phi}, \quad \mathbf{a}_3^{\tilde{\phi}} = \frac{\partial_1 \tilde{\phi} \times \partial_2 \tilde{\phi}}{\|\partial_1 \tilde{\phi} \times \partial_2 \tilde{\phi}\|}.$$

The quadratic form  $Q_2$  on  $\mathbb{M}^{2 \times 2}$  is defined by

$$Q_2(\mathbf{X}) := \min_{\mathbf{z} \in \mathbb{R}^3} Q_3 \left( \sum_{i,j=1}^2 X_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{z} \otimes \mathbf{e}_3 \right) \quad (3.2.10)$$

where  $Q_3(\mathbf{X}) := \frac{\partial^2}{\partial \mathbf{X}^2} W_2^1(\mathbf{I})(\mathbf{X}, \mathbf{X})$ .

- e) Let  $\alpha \in \langle 2, +\infty \rangle$ , and let  $q = 2$ . Let additionally  $W_{2,+\infty}^1$  satisfies conditions **W2.4** and **W2.6**. Then the  $\Gamma$ -limit of the family of functionals  $(J_h^{3d, \langle 2, +\infty \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$J_0^{3d, \langle 2, +\infty \rangle}(\tilde{\phi}) = \begin{cases} - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & \tilde{\phi} \in \mathcal{B}_0^{3d, \langle 2, +\infty \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (3.2.11)$$

where

$$\mathcal{B}_0^{3d, \langle 2, +\infty \rangle} := \{ \tilde{\phi} \in C(\omega; \mathbb{R}^3) : \exists \mathbf{R} \in \text{SO}(3), \mathbf{d} \in \mathbb{R}^3 \text{ s.t. } \tilde{\phi}(\mathbf{x}') = \mathbf{R} \begin{bmatrix} \mathbf{x}' & 0 \end{bmatrix}^T + \mathbf{d} \}. \quad (3.2.12)$$

The claim **b**) is proved in [52], Theorem 2 (under the additional assumption **W2.1**) and in [7], Theorem 1 (under the additional assumption **W2.2** and **W2.3**). The claim **c**) is proved in [30], Theorem 1.1. and (1.6). The claim **d**) is proved in [38], Theorem 6.1. In Section 3.3 we will give proofs for claims **a**) and **e**).

**Remark 3.2.2.** By observing limits of thin 3d bodies obtained in Theorem 3.2.1 in various regimes, we can compare their properties to the limits of thin parts of 3d–thin 3d structures we obtained in the linear case in Theorem 2.2.1 and explained in Remark 2.2.2. For  $\alpha = 2$  we obtain the nonlinear flexural plate model. Similarly as in the linear case, it penalizes the appropriate bending energy in the set of inextensible deformations. For  $\alpha > 2$  we obtain that the model allows only rigid transformation (i.e. the layer is so stiff that allows only rotations and translations). Key differences to linear case for thin layer are that mentioned results from Theorem 3.2.1 correspond to linear models from Theorem 2.2.1, but are not the same due to linearizations in Part I. For  $\alpha < 0$  we again get no contribution of the thin layer, neither in the energy nor in the set of admissible functions.

The largest differences between linear and nonlinear modelling are seen in cases  $\alpha \in [0, 2)$ . Firstly, there are no results in literature that prove rigorous asymptotic derivation

of the model in the case  $\alpha \in [5/3, 2]$ . Up to the today's knowledge, it is an open problem. For that reason Theorem 3.2.1 lacks such result. Also, all similar convergence results in Part II of the thesis will again lack a result in the case  $\alpha \in [5/3, 2]$ . In the case  $\alpha = 0$  in the limit we obtain the nonlinear membrane plate model. In the case  $\alpha \in \langle 0, 5/3 \rangle$  we obtain the so called constrained membrane plate model. It has no contribution in the energy, only a constraint in the set of admissible functions that allows only short maps. It is different to the linear case  $\alpha \in \langle 0, 2 \rangle$  where only (linearly) inextensible deformations were allowed. Here one can see the main difference between linear and nonlinear modelling of thin elastic structures.

Under certain assumptions, in the membrane plate energy short maps do not contribute in energy as well, i.e. nonlinear membrane plates offer no resistance to crumpling. To quote Le Dret and Raoult ([52]), "This is an empirical fact, witnessed by anyone who ever played with a deflated balloon". Indeed, for a thin membrane as a deflated balloon one can compress the body with investing (almost) zero energy. On the other hand, a balloon does resist when stretched. This makes the nonlinear theory more precise for modelling real life examples, since in the linear theory both stretching and compressing are equally penalized in the membrane energy.



### 3.3. TECHNICAL RESULTS AND PROOFS

Before proving the two remaining parts from Theorem 3.2.1, we need a coercivity lemma we are going to use in this proofs and later on in the thesis.

**Lemma 3.3.1.** Let  $\mathbf{f} \in L^{q'}(\Gamma^1; \mathbb{R}^3)$  and  $\alpha \in \langle -1, +\infty \rangle$ . Let  $W_\alpha^1$  satisfies assumptions W1.1), W1.2) and W1.3). Then there are constants  $c > 0$ ,  $C \in \mathbb{R}$  such that the inequality

$$\begin{aligned} & \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + \frac{1}{h^\alpha} \int_{\Omega^1} W_\alpha^1(\nabla_h \phi) d\mathbf{x} - \int_{\Gamma^1} \mathbf{f} \cdot \phi d\mathbf{x}' \\ & \geq c \left( \frac{1}{h^{q-1}} \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + \chi_{\{\alpha \geq 0\}} \|\phi\|_{W^{1,q}(\Omega^1; \mathbb{R}^3)}^q + \|\phi\|_{L^q(\Gamma^0; \mathbb{R}^3)}^q \right) - C \end{aligned} \quad (3.3.1)$$

holds for all  $\phi \in W^{1,q}(\Omega^1; \mathbb{R}^3)$  and all  $h \in \langle 0, 1 \rangle$ .

*Proof.* Let us define  $h_\alpha := h^{\max\{0, \alpha\}}$  and  $h_{-\alpha} := h^{\max\{0, -\alpha\}}$ . Note that  $h_\alpha \leq 1$  and  $h_{-\alpha} \leq 1$ . Also:  $\frac{h_\alpha}{h^\alpha} = h^{\max\{-\alpha, 0\}} = h_{-\alpha}$ .

Since  $W_\alpha^1$  is nonnegative and coercive (from W1.1) and W1.3)), for each  $\lambda \in [0, 1]$  it holds

$$W_\alpha^1(\mathbf{X}) \geq \lambda(c_1 \|\mathbf{X}\|_F^q - c_2).$$

We apply this inequality for  $\lambda = h_\alpha$  and obtain:

$$\frac{1}{h^\alpha} \int_{\Omega^1} W_\alpha^1(\nabla_h \phi) d\mathbf{x} \geq c_1 h_{-\alpha} \|\nabla' \phi\|_{L^q(\Omega^1; \mathbb{R}^{3 \times 2})}^q + \frac{c_1 h_{-\alpha}}{h^q} \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q - c_2 h_{-\alpha}. \quad (3.3.2)$$

Since  $h_{-\alpha} \leq 1$ , the last term can be estimated from below by  $-c_2$ .

Let us now take any nonnegative  $\eta \in W^{1,1}(\Omega^1; \mathbb{R})$ . For a.e.  $\mathbf{x} = (\mathbf{x}', x_3) \in \Omega^1$  we have

$$\eta(\mathbf{x}', 1) \leq \eta(\mathbf{x}', x_3) + \left| \int_{x_3}^1 \partial_3 \eta(\mathbf{x}', y_3) dy_3 \right| \leq \eta(\mathbf{x}', x_3) + \int_0^1 |\partial_3 \eta(\mathbf{x}', y_3)| dy_3. \quad (3.3.3)$$

After integrating over  $\mathbf{x}' \in \omega$  and then  $x_3 \in [0, 1]$  we obtain

$$\int_{\Gamma^1} \eta d\mathbf{x}' \leq \int_{\Omega^1} \eta d\mathbf{x} + \int_{\Omega^1} |\partial_3 \eta| d\mathbf{x}. \quad (3.3.4)$$

We will use this inequality for  $\eta(\mathbf{x}) = |\phi(\mathbf{x})|^q$ . For its derivative we use Young's inequality

$$|\partial_3 \eta(\mathbf{x})| \leq q |\phi|^{q-1} |\partial_3 \phi| \leq (q-1) |\phi|^q + |\partial_3 \phi|^q \quad (3.3.5)$$

to obtain

$$\|\phi\|_{L^q(\Gamma^1; \mathbb{R}^3)}^q \leq q \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q. \quad (3.3.6)$$

From (3.3.2), (3.3.6) and Hölder inequality we obtain

$$\begin{aligned} & \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + \frac{1}{h^\alpha} \int_{\Omega^1} W_\alpha^1(\nabla_h \phi) d\mathbf{x} - \int_{\Gamma^1} \mathbf{f} \cdot \phi d\mathbf{x}' \\ & \geq \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + c_1 h_{-\alpha} \|\nabla' \phi\|_{L^q(\Omega^1; \mathbb{R}^{3 \times 2})}^q \\ & \quad + \frac{c_1 h_{-\alpha}}{h^q} \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q - c_2 - M \|\phi\|_{L^q(\Gamma^1; \mathbb{R}^3)} \\ & \geq \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + c_1 h_{-\alpha} \|\nabla' \phi\|_{L^q(\Omega^1; \mathbb{R}^{3 \times 2})}^q + \frac{c_1 h_{-\alpha}}{h^q} \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q - c_2 - M'I, \end{aligned} \quad (3.3.7)$$

where

$$I = \left( \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q \right)^{1/q}. \quad (3.3.8)$$

Note that from  $\alpha > -1$ , we have  $\max\{0, -\alpha\} < 1$ , so  $\frac{h_{-\alpha}}{h^q} \geq \frac{1}{h^{q-1}}$ . Also,  $\frac{1}{h^{q-1}} \geq 1$ .

Combining these inequalities with conclusion in (3.3.7), we obtain

$$\begin{aligned} & \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + \frac{1}{h^\alpha} \int_{\Omega^1} W_\alpha^1(\nabla_h \phi) d\mathbf{x} - \int_{\Gamma^1} \mathbf{f} \cdot \phi d\mathbf{x}' \\ & \geq \frac{1}{2} \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + c_1 h_{-\alpha} \|\nabla' \phi\|_{L^q(\Omega^1; \mathbb{R}^{3 \times 2})}^q + \frac{c_1}{2h^{q-1}} \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q - c_2 \\ & \quad + \frac{1}{2} \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + \frac{c_1}{2h^{q-1}} \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q - M'I \\ & \geq \frac{1}{2} \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + c_1 h_{-\alpha} \|\nabla' \phi\|_{L^q(\Omega^1; \mathbb{R}^{3 \times 2})}^q + \frac{c_1}{2h^{q-1}} \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q - c_2 \\ & \quad + (mI^q - M'I), \end{aligned} \quad (3.3.9)$$

with  $m = \min\{\frac{1}{2}, \frac{c_1}{2}\}$ . Since the function  $x \mapsto ax^q + bx$  (for  $a > 0, b \in \mathbb{R}$ ) is bounded from below on  $x \in [0, +\infty)$ , a direct consequence is that

$$\begin{aligned} & \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + \frac{1}{h^\alpha} \int_{\Omega^1} W_\alpha^1(\nabla_h \phi) d\mathbf{x} - \int_{\Gamma^1} \mathbf{f} \cdot \phi d\mathbf{x}' \\ & \geq \frac{1}{2} \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + c_1 h_{-\alpha} \|\nabla' \phi\|_{L^q(\Omega^1; \mathbb{R}^{3 \times 2})}^q + \frac{c_1}{2h^{q-1}} \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q - c'_2 \\ & \geq c'_1 \left( \frac{1}{h^{q-1}} \|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + \chi_{\{\alpha \geq 0\}} \|\nabla' \phi\|_{L^q(\Omega^1; \mathbb{R}^{3 \times 2})}^q + \|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q \right) - c'_2. \end{aligned} \quad (3.3.10)$$

To conclude inequality (3.3.1) we need two final observations. Firstly, in the same manner we obtained the inequality (3.3.6), we can bound the term  $\|\phi\|_{L^q(\Gamma^0; \mathbb{R}^3)}$  by  $\|\partial_3 \phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q$

and  $\|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q$  and adjusting constants (note that  $\frac{1}{h^{q-1}} \geq 1$ ). Secondly, when  $\alpha \geq 0$ , on the right hand side we have all derivatives of  $\phi$ , so we have bounded the term  $\|\nabla \phi\|_{L^q(\Omega^1; \mathbb{R}^{3 \times 3})}^q$ .  $\square$

We now prove the two remaining parts of Theorem 3.2.1.

*Proof of Theorem 3.2.1.a).* Let us prove the claim by the definition. For the "lim inf" part, let us take any  $\phi \in L^q(\Omega^1; \mathbb{R}^3)$  and any  $(\phi_h)_{h>0} \subset L^q(\Omega^1; \mathbb{R}^3)$  converging strongly to  $\phi$  in  $L^q(\Omega^1; \mathbb{R}^3)$ . We have to prove that  $\partial_3 \phi = 0$  and  $\liminf_{h \rightarrow 0} J_h^{3d, \langle -1, 0 \rangle}(\phi_h) \geq J_0^{3d, \langle -1, 0 \rangle}(\tilde{\phi})$ . We can firstly take subsequence of  $(\phi_h)_{h>0}$  (without changing its notation) such that  $\liminf_{h \rightarrow 0} J_h^{3d, \langle -1, 0 \rangle}(\phi_h) = \lim_{h \rightarrow 0} J_h^{3d, \langle -1, 0 \rangle}(\phi_h) =: L$ . If  $L = +\infty$ , we have nothing to prove. Thus in the sequel we assume  $L < +\infty$  and consequently there is a constant  $C \in \mathbb{R}$  such that  $J_h^{3d, \langle -1, 0 \rangle}(\phi_h) < C$  for all  $h > 0$ . Due to strong convergence of  $(\phi_h)_{h>0}$ , we know that  $\|\phi_h\|_{L^q(\Omega^1; \mathbb{R}^3)}$  is uniformly bounded. By applying Lemma 3.3.1, we obtain that  $\partial_3 \phi_h \rightarrow 0$ . Due to uniqueness of limits in distributions, we obtain that  $\partial_3 \phi = 0$ , and thus for the limit  $\phi$  we have  $J_0^{3d, \langle -1, 0 \rangle}(\phi) < +\infty$ . Now, since the first part of functionals  $J_h^{3d, \langle -1, 0 \rangle}$  is nonnegative and the second one is linear, we clearly have the desired inequality  $\liminf_{h \rightarrow 0} J_h^{3d, \langle -1, 0 \rangle}(\phi_h) \geq J_0^{3d, \langle -1, 0 \rangle}(\tilde{\phi})$ .

Let us now prove the "lim sup" part of the definition. Since the part  $\tilde{\phi} \mapsto \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}'$  is linear, it is continuous with respect to the convergence in  $L^q(\omega; \mathbb{R}^3)$ , so it does not affect the convergence so without loss of generality  $\mathbf{f} = 0$ . Let us take arbitrary  $\tilde{\phi} \in L^q(\omega; \mathbb{R}^3)$ . Let us take any  $(\tilde{\phi}_n)_{n \geq 1} \subset C^\infty(\omega; \mathbb{R}^3)$  that strongly converges to  $\tilde{\phi}$  in  $L^q(\omega; \mathbb{R}^3)$ .

Let us define

$$n(h) := \arg \max_{n \geq 1} \{ \|\nabla \tilde{\phi}_n\|_{L^q(\omega; \mathbb{R}^{3 \times 3})} < h^{\alpha/(2q)} \}, \quad (3.3.11)$$

in case that  $\|\nabla \tilde{\phi}_n\|_{L^q(\omega; \mathbb{R}^{3 \times 3})}$  is not uniformly bounded (note that  $h^{\alpha/(2q)} \rightarrow +\infty$  as  $h \rightarrow 0$ , so the mapping is well defined for all sufficiently small values of  $h$ ), and  $n(h) = \lceil 1/h \rceil$  if  $\|\nabla \tilde{\phi}_n\|_{L^q(\omega; \mathbb{R}^{3 \times 3})}$  is uniformly bounded.

If  $\|\nabla \tilde{\phi}_n\|_{L^q(\omega; \mathbb{R}^{3 \times 3})}$  is uniformly bounded, then clearly  $n(h) \rightarrow +\infty$  and due to  $h^{\alpha/(2q)} \rightarrow +\infty$  the bound  $\|\nabla \tilde{\phi}_{n(h)}\|_{L^q(\omega; \mathbb{R}^{3 \times 3})} < h^{\alpha/(2q)}$  is satisfied for sufficiently small values of  $h$ . If  $\|\nabla \tilde{\phi}_n\|_{L^q(\omega; \mathbb{R}^{3 \times 3})}$  is not uniformly bounded, as  $h \rightarrow 0$ , due to  $h^{\alpha/(2q)} \rightarrow +\infty$  we have  $n(h) \rightarrow +\infty$ . In both cases we have that  $(\phi_h)_{h>0}$ , defined as  $\phi_h := \tilde{\phi}_{n(h)}$ ,

converges to  $\phi$  in  $L^q(\Omega^1; \mathbb{R}^3)$ . From the growth condition W2.1) for sufficiently small values of  $h$  we obtain

$$0 \leq J_h^{3d, \langle -1, 0 \rangle}(\phi_h) \leq \frac{1}{h^\alpha} C \left( 1 + \|\nabla' \phi_h\|_{L^p(\omega; \mathbb{R}^{3 \times 2})}^q \right) \leq \frac{1}{h^\alpha} C \left( 1 + 2^q h^{\alpha/2} \right) \rightarrow 0. \quad (3.3.12)$$

This proves that  $\limsup J_h^{3d, \langle -1, 0 \rangle}(\phi_h) = 0 = J_0^{3d, \langle -1, 0 \rangle}(\tilde{\phi})$ .  $\square$

*Proof of Theorem 3.2.1.e).* Let us firstly prove the lower bound. Let  $(J_h^{3d, 2})_{h>0}$ ,  $J_0^{3d, 2}$  and  $\mathcal{B}_0^{3d, 2}$  be the energy functionals for  $h > 0$ , its  $\Gamma$ -limit and the set of admissible functions from Theorem 3.2.1.d). Since assumptions in Theorem 3.2.1.d) coincide with assumptions in Theorem 3.2.1.e), we are able to use mentioned objects. Since now  $\alpha > 2$ , we have  $J_h^{3d, \langle 2, +\infty \rangle}(\phi) \geq J_h^{3d, 2}(\phi)$ , so the same holds for their  $\Gamma$ -limits and the set of admissible functions of  $\Gamma$ -limit of  $J_h^{3d, \langle 2, +\infty \rangle}(\phi)$  is necessarily a subset of  $\mathcal{B}_0^{3d, 2}$ . Now, without loss of generality let  $f = 0$  (since the linear part will not affect the  $\Gamma$ -limit). Note that now all functionals are nonnegative.

Let us take any  $\tilde{\phi} \in \mathcal{B}_0^{3d, 2}$  and any  $(\phi_h)_{h>0} \subset W^{1, 2}(\Omega^1; \mathbb{R}^3)$  that strongly converges to  $\phi$  in  $L^2(\Omega^1; \mathbb{R}^3)$  and such that values  $J_h^{3d, \langle 2, +\infty \rangle}(\phi_h)$  are uniformly bounded by  $C > 0$ . If there does not exist such sequence, there is nothing to prove, the "lim inf" inequality is trivially satisfied. For those functions we have  $\liminf_{h \rightarrow 0} J_h^{3d, 2}(\phi_h) \geq J_0^{3d, 2}(\tilde{\phi})$ , so there exists a subsequence of  $(\phi_h)_{h>0}$  (still denoted the same) such that

$$(\forall \delta > 0) (\exists h_0 > 0) (\forall h \in \langle 0, h_0 \rangle) J_h^{3d, 2}(\phi_h) \geq J_0^{3d, 2}(\tilde{\phi}) - \delta. \quad (3.3.13)$$

Let us take  $\delta = \frac{1}{2} J_0^{3d, 2}(\tilde{\phi})$  and multiply the last inequality by  $h^{2-\alpha}$ . Then we have

$$C \geq J_h^{3d, \langle 2, +\infty \rangle}(\phi_h) \geq \frac{1}{2h^{\alpha-2}} J_0^{3d, 2}(\tilde{\phi}), \quad (3.3.14)$$

so when  $h \rightarrow 0$  we obtain that necessarily  $J_0^{3d, 2}(\tilde{\phi}) = 0$ . From Lemma 3.3.2 we obtain that  $\tilde{\phi}$  is rigid transformation. Now the lower bound is clear.

For the "lim sup" part, let us take any  $\tilde{\phi} \in \mathcal{B}_0^{3d, \langle 2, +\infty \rangle}$ . Let us define

$$\phi_h(x) := \tilde{\phi}(x') + h(\partial_1 \tilde{\phi} \times \partial_2 \tilde{\phi})x_3. \quad (3.3.15)$$

Since  $\nabla_h \phi_h$  is a constant rotation from  $\text{SO}(3)$  (for which  $W_{\langle 2, +\infty \rangle}^1$  is equal to zero by the assumptions), and since clearly  $\phi_h \rightarrow \phi$  in  $L^2(\Omega^1; \mathbb{R}^3)$ , the claim is proven.  $\square$

**Lemma 3.3.2.** Let  $\tilde{\phi} \in W^{2,2}(\omega; \mathbb{R}^3)$  such that  $\int_{\Omega^1} Q_2(\mathbf{b}(\tilde{\phi})) d\mathbf{x}' = 0$ . Then  $\tilde{\phi}$  is rigid transformation.

*Proof.* Since  $Q_2$  is coercive quadratic form, we have that  $\mathbf{b}(\tilde{\phi}) = 0$  a.e.,  $b_{\beta_1, \beta_2}(\tilde{\phi}) = \partial_{\beta_1} \mathbf{a}_3^{\tilde{\phi}} \cdot \partial_{\beta_2} \tilde{\phi}$ ,  $\mathbf{a}_3^{\tilde{\phi}} = \frac{\partial_1 \tilde{\phi} \times \partial_2 \tilde{\phi}}{\|\partial_1 \tilde{\phi} \times \partial_2 \tilde{\phi}\|}$ . We see that  $\partial_{\beta} \mathbf{a}_3^{\tilde{\phi}}$  is perpendicular to  $\partial_1 \tilde{\phi}$  and  $\partial_2 \tilde{\phi}$ . By differentiating the equality  $\|\mathbf{a}_3^{\tilde{\phi}}\|^2 = 1$ , we obtain  $\partial_{\beta} \mathbf{a}_3^{\tilde{\phi}} \cdot \mathbf{a}_3^{\tilde{\phi}} = 0$ , so specially  $\partial_{\beta} \mathbf{a}_3^{\tilde{\phi}}$  is perpendicular to all vectors in the basis  $\{\partial_1 \tilde{\phi}, \partial_2 \tilde{\phi}, \mathbf{a}_3^{\tilde{\phi}}\}$ , so  $\partial_{\beta} \mathbf{a}_3^{\tilde{\phi}} = 0$ ,  $\beta = 1, 2$ . Specially,  $\mathbf{a}_3^{\tilde{\phi}}$  is a constant.

By differentiating  $\|\partial_1 \tilde{\phi}\|^2 = 1$  we obtain  $\partial_{12} \tilde{\phi} \cdot \partial_1 \tilde{\phi} = 0$ . Similarly  $\partial_{12} \tilde{\phi} \cdot \partial_2 \tilde{\phi} = 0$ . Lastly, from definition of  $\mathbf{a}_3^{\tilde{\phi}}$  we have  $\partial_{\beta} \tilde{\phi} \cdot \mathbf{a}_3^{\tilde{\phi}} = 0$ , so by differentiating this and using  $\mathbf{a}_3^{\tilde{\phi}} = \text{const}$  we obtain  $\partial_{\beta_1, \beta_2} \tilde{\phi} \cdot \mathbf{a}_3^{\tilde{\phi}} = 0$ ,  $\beta_1, \beta_2 = 1, 2$ . Thus,  $\partial_{12} \tilde{\phi}$  is also perpendicular to all vectors in the basis, and thus  $\partial_{12} \tilde{\phi} = 0$ .

We again differentiate  $\|\partial_1 \tilde{\phi}\|^2 = 1$  and  $\partial_1 \tilde{\phi} \cdot \partial_2 \tilde{\phi} = 0$  and use obtained conclusions to get  $\partial_{11} \tilde{\phi} \cdot \partial_1 \tilde{\phi} = \partial_{11} \tilde{\phi} \cdot \partial_2 \tilde{\phi} = 0$ . From before we know that  $\partial_{\beta_1, \beta_2} \tilde{\phi}$  are perpendicular to  $\mathbf{a}_3^{\tilde{\phi}}$ , so again by the same argument  $\partial_{11} \tilde{\phi} = 0$ . After applying analogous arguments for  $\partial_{22} \tilde{\phi}$ , we obtain  $D^2 \tilde{\phi} = 0$ , thus  $\tilde{\phi}$  is affine function. Due to conditions  $\|\partial_1 \tilde{\phi}\| = \|\partial_2 \tilde{\phi}\| = 1$  and  $\partial_1 \tilde{\phi} \cdot \partial_2 \tilde{\phi} = 0$ , we obtain that it is necessarily a rigid transformation.  $\square$

**Remark 3.3.3.** All the parts in Theorem 3.2.1 are proved by the definition of the  $\Gamma$ -convergence, which included the "lim sup" part of the definition: the step in which for arbitrary admissible  $\phi \in L^q(\Omega^1; \mathbb{R}^3)$  the family  $(\phi_h)_{h>0} \subset L^q(\Omega^1; \mathbb{R}^3)$  is obtained such that  $\phi_h \rightarrow \phi$  in  $L^q(\Omega^1; \mathbb{R}^3)$  and

$$\limsup_{h \rightarrow 0} J_h^{3d, \alpha}(\phi_h) \leq J_0^{3d, \alpha}(\phi).$$

The properties of these families  $(\phi_h)_{h>0}$  is going to play important role in the rest of this thesis, so we are now going to get into more details about those families.

- a) For  $\alpha \in \langle -1, 0 \rangle$ , from the proof of Theorem 3.2.1.a) we see that the family is obtained by smooth approximations of the function  $\tilde{\phi} \in L^q(\omega; \mathbb{R}^3)$ . However, for  $\tilde{\phi} \in W^{1,q}(\omega; \mathbb{R}^3)$  the proof would hold for the constant family  $\tilde{\phi}_h = \tilde{\phi}$ :

$$\begin{aligned} J_h^{3d, \langle -1, 0 \rangle}(\phi_h) &\leq \frac{1}{h^\alpha} C \left( 1 + \|\nabla' \tilde{\phi}\|_{L^q(\omega; \mathbb{R}^{3 \times 2})}^q \right) - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} \\ &\rightarrow - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} = J_0^{3d, \langle -1, 0 \rangle}(\tilde{\phi}). \end{aligned} \quad (3.3.16)$$

- b) For  $\alpha = 0$ , from [52] (Proposition 7., (24)) and [7] (Theorem 1) we see that the "lim sup" part of the proof is obtained by the family of the form  $\phi_h = \tilde{\phi}^0 + h(x_3 - 1/2)\tilde{\phi}^1$  for some  $\tilde{\phi}^0, \tilde{\phi}^1 \in W^{1,q}(\omega; \mathbb{R}^3)$ ,  $h > 0$ .
- c) For  $\alpha \in \langle 0, 5/3 \rangle$ , from [30] (Lemma 5.1.) we see that the "lim sup" part of the proof is obtained by the family of the form  $\phi_h = \tilde{\phi}_h^0 + hx_3\tilde{\phi}_h^1$ , where  $\tilde{\phi}_h^0, \tilde{\phi}_h^1 \in W^{1,2}(\omega; \mathbb{R}^3)$ ,  $h > 0$ .
- d) For  $\alpha = 2$ , from [38] (Theorem 6.1. and (6.24.)) we see that the "lim sup" part of the proof is obtained by the family of the form  $\phi_h = \tilde{\phi}_h^0 + h(x_3 - 1/2)\tilde{\phi}_h^1 + h^2 \frac{(x_3 - 1/2)^2}{2} \tilde{\phi}_h^2$ , where  $\tilde{\phi}_h^0, \tilde{\phi}_h^1, \tilde{\phi}_h^2 \in W^{1,2}(\omega; \mathbb{R}^3)$ ,  $h > 0$ .
- e) For  $\alpha \in \langle 2, +\infty \rangle$ , from (3.3.15) in the proof of Theorem 3.2.1.e) we see that the family is of the form  $\phi_h = \tilde{\phi}^0 + hx_3h\tilde{\phi}^1$  for some  $\tilde{\phi}^0, \tilde{\phi}^1 \in W^{1,2}(\omega; \mathbb{R}^3)$ ,  $h > 0$ .

## 4. NONLINEAR 3D–THIN 3D MODEL

### 4.1. PROBLEM SETUP

In this chapter we are going to observe the same problem as in Chapter 2, but in terms of the nonlinear modelling.

Let us define sets

$$\begin{aligned}
 \omega &= \langle 0, 1 \rangle \times \langle 0, 1 \rangle, \\
 \Omega^0 &= \omega \times \langle -1, 0 \rangle, \\
 \Omega^h &= \omega \times \langle 0, h \rangle, \\
 \Omega^{0+h} &= \omega \times \langle -1, h \rangle, \\
 \Gamma^s &= \omega \times \{s\}, \text{ for } s \in \mathbb{R}.
 \end{aligned} \tag{4.1.1}$$

We consider  $\overline{\Omega^{0+h}} = \overline{\Omega^0 \cup \Omega^h}$  to be an elastic body made of two materials with possibly different properties at different parts of the domain ( $\Omega^0$  and  $\Omega^h$ ), see Figure 4.1. The stored energy function for  $\Omega^0$  is given by  $W^0 : \mathbb{M}^{3 \times 3} \rightarrow \mathbb{R}$  and the stored energy function for  $\Omega^1$  is given by  $W^1 : \mathbb{R} \times \mathbb{M}^{3 \times 3} \rightarrow \mathbb{R}$  (it also depends on the thickness parameter  $h$ ). Let  $\bar{\phi} : \Omega^h \rightarrow \mathbb{R}$  be a function describing the position  $\bar{\phi}(\mathbf{x})$  of the point  $\mathbf{x} \in \Omega^h$  after deformation of the body; similarly we define function  $\psi : \Omega^0 \rightarrow \mathbb{R}$ . Further we assume that the body is clamped at  $x_3 = -1$ , that a contact force  $\mathbf{f}$  is applied at the boundary  $\Gamma^h = \omega \times \{h\}$ , and that the remaining boundary is force free. For simplicity we assume that there are no body forces applied.

Analogously as in Chapter 3, we perform the rescaling of the thin part by the function

$$(\mathbf{x}', x_3) \in \Omega^1 \mapsto (\mathbf{x}', hx_3) \in \Omega^h,$$

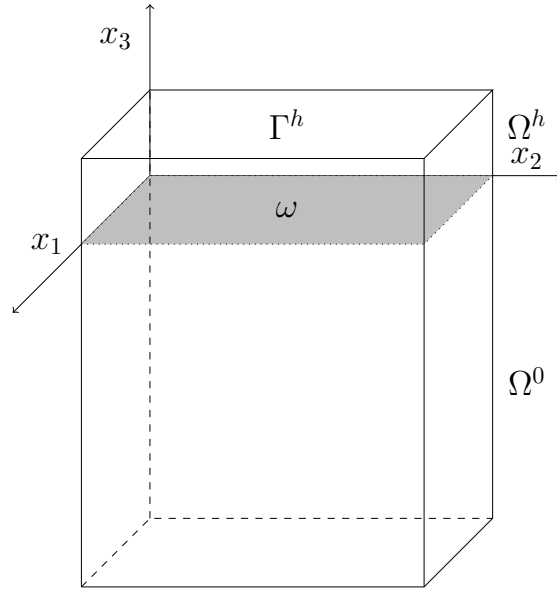


Figure 4.1: 3d elastic body.

so that problems for all  $h > 0$  are defined on the same domain  $\Omega^{0+1}$ . With the same reasoning in the Section 3.1, in the thin part of the domain we also incorporate the behaviour of elasticity coefficients in  $W^1$  in such way that there is a function  $W_\alpha^1 : \mathbb{M}^{3 \times 3} \rightarrow \mathbb{R}$ , independent of  $h$ , such that

$$W^1(h; \mathbf{X}) = \frac{1}{h^{\alpha+1}} W_\alpha^1(\mathbf{X}) \quad (4.1.2)$$

for all  $h > 0$ ,  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ , and for a real parameter  $\alpha > -1$ .

Now the total energy functional for the thin part of the structure from the Figure 4.1 on the rescaled domain  $\Omega^1$  is given by the functional  $J_h^{3d,\alpha} : L^q(\Omega^1; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  defined in (3.1.2).

For  $p \in \langle 1, +\infty \rangle$  let us define the total energy functional for the elastic body situated in  $\Omega^0$  as  $\mathcal{W}^0(\psi) : L^p(\Omega^0; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  given by

$$\mathcal{W}^0(\psi) = \begin{cases} \int_{\Omega^0} W^0(\nabla \psi) dx & \psi \in W^{1,p}(\Omega^0; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (4.1.3)$$

The functional  $\mathcal{W}^0$  is independent of  $h$ , and so its elasticity coefficients are.

Then the total energy functional for the whole structure is defined as the sum of those two functionals up to a definition of the set of admissible functions. For arbitrary  $\alpha > -1$



and  $h > 0$  we define  $K_h^{3d,\alpha} : L^p(\Omega^0; \mathbb{R}^3) \times L^q(\Omega^1; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  by

$$K_h^{3d,\alpha}(\psi, \phi) = \begin{cases} \int_{\Omega^0} W^0(\nabla \psi) d\mathbf{x} \\ \quad + \frac{1}{h^\alpha} \int_{\Omega^1} W_\alpha^1(\nabla_h \phi) d\mathbf{x} - \int_{\Gamma^1} \mathbf{f} \cdot \phi d\mathbf{x} & (\psi, \phi) \in \mathcal{A}^{3d,\alpha}, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.1.4)$$

where

$$\begin{aligned} \mathcal{A}^{3d,\alpha} := \{(\psi, \phi) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{1,q}(\Omega^1; \mathbb{R}^3) : \\ \psi|_{x_3=0} = \phi|_{x_3=0}, \psi|_{x_3=-1} = \text{id}\}. \end{aligned} \quad (4.1.5)$$

We are interested in obtaining  $\Gamma$ –limits of the families of  $(K_h^{3d,\alpha})_{h>0}$ , for various  $\alpha > -1$ .

For the  $W_\alpha^1$  and  $\mathbf{f} \in L^{q'}(\Gamma^1; \mathbb{R}^3)$ ,  $q' = (1 - 1/q)^{-1}$  we assume the same conditions [W1.1](#)), [W1.2](#)) and [W1.3](#)) as in Chapter 3. Additionally, for  $W^0$  we assume

W0.1)  $1 < p \leq q < +\infty$ ,

W0.2)  $W^0 : \mathbb{M}^{3 \times 3} \rightarrow [0, +\infty)$  is continuous,

W0.3)  $W^0$  satisfies the coercivity condition:  $W^0(\mathbf{X}) \geq c_1 \|\mathbf{X}\|_F^p - c_2$ , for  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ , for real constants  $c_1 > 0, c_2$ ,

W0.4) there exists  $C > 0$  such that  $W^0(\mathbf{X}) \leq C(1 + \|\mathbf{X}\|_F^p)$ , for all  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ .

Again, for  $W^0$  we didn't assume that it is quasiconvex. It will not cause us problems in our work due to reasons explained in the Remark [3.1.1](#). If  $QW^0$  is its quasiconvex envelope, then

$$\Gamma - (\mathcal{W}^0(\psi)) = \int_{\Omega^0} QW^0(\nabla \psi).$$

## 4.2. THE MAIN RESULT

In this section we are going to determine the  $\Gamma$ –limit of families  $(K_h^{3d,\alpha})_{h>0}$  for different values of  $\alpha > -1$ .

We again use the *tilde* notation and isomorphisms

$$\{\phi \in L^q(\Omega^1; \mathbb{R}^3) : \partial_3 \phi = 0\} \equiv \{\tilde{\phi} \in L^q(\omega; \mathbb{R}^3)\}, \quad (4.2.1)$$

and

$$\{\phi \in W^{1,q}(\Omega^1; \mathbb{R}^3) : \partial_3 \phi = 0\} \equiv \{\tilde{\phi} \in W^{1,q}(\omega; \mathbb{R}^3)\}. \quad (4.2.2)$$

**Theorem 4.2.1.** Let the family of functionals  $(K_h^{3d,\alpha})_{h>0}$  be as defined in (4.1.4). Let us assume that conditions **W1.1)–W1.3)** and **W0.1)–W0.4)** are satisfied.

- a) Let  $\alpha \in \langle -1, 0 \rangle$ . Let additionally  $W_{\langle -1, 0 \rangle}^1$  satisfies the condition **W2.1)**. Then the  $\Gamma$ –limit of the family of functionals  $(K_h^{3d,\langle -1, 0 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$K_0^{3d,\langle -1, 0 \rangle}(\psi, \tilde{\phi}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & (\psi, \tilde{\phi}) \in \mathcal{A}_0^{3d,\langle -1, 0 \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.2.3)$$

where

$$\mathcal{A}_0^{3d,\langle -1, 0 \rangle} = \{(\psi, \tilde{\phi}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times L^q(\omega; \mathbb{R}^3) : \psi|_{x_3=0} = \tilde{\phi}, \psi|_{x_3=-1} = \text{id}\}. \quad (4.2.4)$$

- b) Let  $\alpha = 0$ . Let additionally  $W_0^1$  satisfies either the condition **W2.1)**, or conditions **W2.2)** and **W2.3)**. Then the  $\Gamma$ –limit of the family of functionals  $(K_h^{3d,0})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$K_0^{3d,0}(\psi, \tilde{\phi}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} \\ + \int_{\omega} Q\tilde{W}_0^1(\nabla' \tilde{\phi}) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & (\psi, \tilde{\phi}) \in \mathcal{A}_0^{3d,0}, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.2.5)$$

where

$$\mathcal{A}_0^{3d,0} = \{(\psi, \tilde{\phi}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{1,q}(\omega; \mathbb{R}^3) : \psi|_{x_3=0} = \tilde{\phi}, \psi|_{x_3=-1} = \text{id}\} \quad (4.2.6)$$

and  $\tilde{W}_0^1(\mathbf{X})$  is from (3.2.5).

- c) Let  $\alpha \in \langle 0, 5/3 \rangle$ , and let  $q = 2$ . Let additionally  $W_{\langle 0, 5/3 \rangle}^1$  satisfies conditions **W2.4** and **W2.5**. Then the  $\Gamma$ –limit of the family of functionals  $(K_h^{3d, \langle 0, 5/3 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$K_0^{3d, \langle 0, 5/3 \rangle}(\psi, \tilde{\phi}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & (\psi, \tilde{\phi}) \in \mathcal{A}_0^{3d, \langle 0, 5/3 \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.2.7)$$

where

$$\mathcal{A}_0^{3d, \langle 0, 5/3 \rangle} = \{(\psi, \tilde{\phi}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{1,\infty}(\omega; \mathbb{R}^3) : \psi|_{x_3=0} = \tilde{\phi}, \psi|_{x_3=-1} = \text{id}, \nabla' \tilde{\phi}^T \nabla' \tilde{\phi} \leq \mathbf{I} \text{ a.e.}\}. \quad (4.2.8)$$

- d) Let  $\alpha = 2$ , and let  $q = 2$ . Let additionally  $W_2^1$  satisfies conditions **W2.4** and **W2.6**. Then the  $\Gamma$ –limit of the family of functionals  $(K_h^{3d,2})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$K_0^{3d,2}(\psi, \tilde{\phi}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} + \frac{1}{24} \int_{\omega} Q_2(\mathbf{b}(\tilde{\phi})) d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & (\psi, \tilde{\phi}) \in \mathcal{A}_0^{3d,2}, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.2.9)$$

where

$$\mathcal{A}_0^{3d,2} = \{(\psi, \tilde{\phi}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{2,2}(\omega; \mathbb{R}^3) : \psi|_{x_3=0} = \tilde{\phi}, \psi|_{x_3=-1} = \text{id}, \nabla \tilde{\phi}^T \nabla \tilde{\phi} = \mathbf{I}\} \quad (4.2.10)$$

where  $\mathbf{b}(\tilde{\phi}) = (b_{\beta_1, \beta_2}(\tilde{\phi}))_{\beta_1, \beta_2}$  is the curvature tensor and  $Q_2$  quadratic form defined in (3.2.10).

e) Let  $\alpha \in \langle 2, +\infty \rangle$ , and let  $q = 2$ . Let additionally  $W_{\langle 2, +\infty \rangle}^1$  satisfies conditions **W2.4** and **W2.6**. Then the  $\Gamma$ –limit of the family of functionals  $(K_h^{3d, \langle 2, +\infty \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^q(\Omega^1; \mathbb{R}^3)$  topology is

$$K_0^{3d, \langle 2, +\infty \rangle}(\psi, \tilde{\phi}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\phi} d\mathbf{x}' & (\psi, \tilde{\phi}) \in \mathcal{A}_0^{3d, \langle 2, +\infty \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.2.11)$$

where

$$\begin{aligned} \mathcal{A}_0^{3d, \langle 2, +\infty \rangle} &= \{(\psi, \tilde{\phi}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times C(\omega; \mathbb{R}^3) : \\ &\quad \psi|_{x_3=0} = \tilde{\phi}, \psi|_{x_3=-1} = \text{id}, \\ &\quad \exists \mathbf{R} \in \text{SO}(3), \mathbf{d} \in \mathbb{R}^3 \text{ s.t. } \tilde{\phi}(\mathbf{x}') = \mathbf{R}(\mathbf{x}', 0) + \mathbf{d}\}. \end{aligned} \quad (4.2.12)$$

### 4.3. TECHNICAL RESULTS

In this section we prove technical results needed for the proof of Theorem 4.2.1. The main part of the proof will be done by Proposition 4.3.1. Its key part of the proof is based on Lemma 4.3.4, which is already proved in [8] as a part of Lemma 2.

For purposes of the next proposition, let us define spaces

$$\begin{aligned} L^{q \times \text{SO}(3)} &:= L^q(\Omega^1; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3)) \\ L^{p \times q \times \text{SO}(3)} &:= L^p(\Omega^0; \mathbb{R}^3) \times L^{q \times \text{SO}(3)}. \end{aligned} \quad (4.3.1)$$

**Proposition 4.3.1.** Let  $1 < p \leq q < \infty$ . Let there be a family of functionals  $(J_h)_{h>0}$ ,  $J_h : L^{q \times \text{SO}(3)} \rightarrow \overline{\mathbb{R}}$  with the set of admissible functions  $\mathcal{B}$ . Let its  $\Gamma$ -limit (in the strong  $L^{q \times \text{SO}(3)}$  topology) be the functional  $J_0$  with the set of admissible functions  $\mathcal{B}_0$ .

Let  $J_h$  satisfy following inequality:

$$\|\phi\|_{L^q(\Omega^1; \mathbb{R}^3)}^q + J_h(\phi, \tilde{\mathbf{S}}) \geq c_J \|\phi\|_{L^p(\Gamma^0; \mathbb{R}^3)}^q - C_J \quad (4.3.2)$$

with  $c_J > 0$ ,  $C_J \in \mathbb{R}$  independent of  $(\phi, \tilde{\mathbf{S}}) \in \mathcal{B}$  and  $h > 0$ .

Let there be a function  $g : \mathbb{M}^{3 \times 3} \rightarrow \mathbb{R}$  satisfying the growth condition  $0 \leq g(\mathbf{X}) \leq C(1 + \|\mathbf{X}\|_F^p)$ ,  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ . Let  $g$  defines a functional  $G : L^p(\Omega^0; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  by

$$G(\psi) = \begin{cases} \int_{\Omega^0} g(\nabla \psi) d\mathbf{x} & \psi \in W^{1,p}(\Omega^0; \mathbb{R}^3), \psi|_{x_3=-1} = \text{id}, \\ +\infty & \text{otherwise.} \end{cases} \quad (4.3.3)$$

Additionally, let  $G$  satisfies the following inequality:

$$\|\psi\|_{L^p(\Omega^0; \mathbb{R}^3)}^p + G(\psi) \geq c_G \left( \|\psi\|_{W^{1,p}(\Omega^0; \mathbb{R}^3)}^p + \|\psi\|_{L^p(\Gamma^{-1}; \mathbb{R}^3)}^p + \|\psi\|_{L^p(\Gamma^0; \mathbb{R}^3)}^p \right) - C_G \quad (4.3.4)$$

with  $c_G > 0$ ,  $C_G \in \mathbb{R}$  independent of  $\psi \in W^{1,p}(\Omega^0; \mathbb{R}^3)$ . Let  $G_0(\psi) := \Gamma - G(\psi)$ .

Let us define a family of functionals  $(K_h)_{h>0}$ ,  $K_h : L^{p \times q \times \text{SO}(3)} \rightarrow \overline{\mathbb{R}}$ ,

$$K_h(\psi, \phi, \tilde{\mathbf{S}}) = \begin{cases} G(\psi) + J_h(\phi, \tilde{\mathbf{S}}) & (\psi, \phi, \tilde{\mathbf{S}}) \in \mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.3.5)$$

where

$$\begin{aligned} \mathcal{A} &:= \{(\psi, \phi, \tilde{\mathbf{S}}) \in L^{p \times q \times \text{SO}(3)} : \\ &\quad \psi \in W^{1,p}(\Omega^0; \mathbb{R}^3), \psi|_{x_3=-1} = \text{id}, \psi|_{x_3=0} = \phi|_{x_3=0}, (\phi, \tilde{\mathbf{S}}) \in \mathcal{B}\}. \end{aligned} \quad (4.3.6)$$

Let us define  $K_0 : L^{p \times q \times \text{SO}(3)} \rightarrow \overline{\mathbb{R}}$  by

$$K_0(\psi, \phi, \tilde{\mathbf{S}}) = \begin{cases} G_0(\psi) + J_0(\phi, \tilde{\mathbf{S}}) & (\psi, \phi, \tilde{\mathbf{S}}) \in \mathcal{A}_0, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.3.7)$$

where

$$\mathcal{A}_0 := \{(\psi, \phi, \tilde{\mathbf{S}}) \in L^{p \times q \times \text{SO}(3)} : \psi \in W^{1,p}(\Omega^0; \mathbb{R}^3), \psi|_{x_3=-1} = \text{id}, \psi|_{x_3=0} = \phi|_{x_3=0}, (\phi, \tilde{\mathbf{S}}) \in \mathcal{B}_0\}. \quad (4.3.8)$$

Then we have the following:

- a) The functional  $K_0$  satisfies the "lim inf" part of the definition of the  $\Gamma$ –limit for  $(K_h)_{h>0}$ : for all  $(\psi, \phi, \tilde{\mathbf{S}}) \in L^{p \times q \times \text{SO}(3)}$  and for all  $(\psi_h, \phi_h, \tilde{\mathbf{S}}_h)_{h>0} \subset L^{p \times q \times \text{SO}(3)}$  converging strongly to  $(\psi, \phi, \tilde{\mathbf{S}})$  in  $L^{p \times q \times \text{SO}(3)}$  we have

$$\liminf_{h \rightarrow 0} K_h(\psi_h, \phi_h, \tilde{\mathbf{S}}_h) \geq K_0(\psi, \phi, \tilde{\mathbf{S}}). \quad (4.3.9)$$

- b) Additionally, let the following condition hold: for all  $(\phi, \tilde{\mathbf{S}}) \in \mathcal{B}_0$  there exists  $(\phi_h, \tilde{\mathbf{S}}_h)_{h>0} \subset \mathcal{B}$  converging strongly to  $(\phi, \tilde{\mathbf{S}})$  in  $L^{q \times \text{SO}(3)}$  such that  $J_h(\phi_h, \tilde{\mathbf{S}}_h) \rightarrow J_0(\phi, \tilde{\mathbf{S}})$  and

$$\|\phi_h\|_{W^{1,q}(\Gamma^0; \mathbb{R}^3)} < C. \quad (4.3.10)$$

Then  $K_0$  satisfies the "lim sup" part of the definition of the  $\Gamma$ –limit for  $(K_h)_{h>0}$ : for all  $(\psi, \phi, \tilde{\mathbf{S}}) \in L^{p \times q \times \text{SO}(3)}$  there exists  $(\psi_h, \phi_h, \tilde{\mathbf{S}}_h)_{h>0} \subset L^{p \times q \times \text{SO}(3)}$  converging strongly to  $(\psi, \phi, \tilde{\mathbf{S}})$  in  $L^{p \times q \times \text{SO}(3)}$  such that

$$\limsup_{h \rightarrow 0} K_h(\psi_h, \phi_h, \tilde{\mathbf{S}}_h) \leq K_0(\psi, \phi, \tilde{\mathbf{S}}). \quad (4.3.11)$$

Specially this implies that  $K_0(\psi, \phi, \tilde{\mathbf{S}})$  is the  $\Gamma$ –limit of  $K_h(\psi, \phi, \tilde{\mathbf{S}})$ , as  $h \rightarrow 0$ .

**Remark 4.3.2.** One can note that in the second part of the proposition (the "lim sup" part) we always know that for all  $(\phi, \tilde{\mathbf{S}}) \in \mathcal{B}_0$  there exists  $(\phi_h, \tilde{\mathbf{S}}_h)_{h>0} \subset \mathcal{B}$  that converges strongly to  $(\phi, \tilde{\mathbf{S}})$  in  $L^{p \times q \times \text{SO}(3)}$  and satisfies  $J_h(\phi_h, \tilde{\mathbf{S}}_h) \rightarrow J_0(\phi, \tilde{\mathbf{S}})$  (from the definition of the  $\Gamma$ –limit of the family  $(J_h)_{h>0}$ ). However, the condition (4.3.10) is additional. We will need to check it separately each time we use the second part of the proposition.

**Remark 4.3.3.** Note that Proposition 4.3.1 can be applied even in following cases:

1°  $J_h$  is independent of  $\tilde{\mathbf{S}}$ , and/or

2°  $J_h$  is defined for  $\tilde{\phi} \in L^q(\omega; \mathbb{R}^3)$  – by including  $\partial_3 \phi = 0$  in  $\mathcal{B}$  and using the isomorphism

$$\{\phi \in L^q(\Omega^1; \mathbb{R}^3) : \partial_3 \phi = 0\} \equiv \{\tilde{\phi} \in L^q(\omega; \mathbb{R}^3)\}.$$

Thus we will be able to use this proposition to determine the  $\Gamma$ –limit of family of functionals  $(K_h^{3d,\alpha})_{h>0}$  from (4.1.4) (by taking into account 1°). The part 2° will be used in following chapters.

*Proof.* Let us prove the first claim. Let us take any  $(\psi, \phi, \tilde{\mathbf{S}}) \in L^{p \times q \times \text{SO}(3)}$  and any  $(\psi_h, \phi_h, \tilde{\mathbf{S}}_h)_{h>0} \subset L^{p \times q \times \text{SO}(3)}$  converging strongly to  $(\psi, \phi, \tilde{\mathbf{S}})$  in  $L^{p \times q \times \text{SO}(3)}$ . Without loss of generality, we can take subsequence of  $(\psi_h, \phi_h, \tilde{\mathbf{S}}_h)_{h>0}$  (without changing its notation) such that

$$\liminf_{h \rightarrow 0} K_h(\psi_h, \phi_h, \tilde{\mathbf{S}}_h) = \lim_{h \rightarrow 0} K_h(\psi_h, \phi_h, \tilde{\mathbf{S}}_h) =: L.$$

If  $L = +\infty$ , we have nothing to prove. Thus in the sequel we assume  $L < +\infty$  and consequently there is a constant  $C \in \mathbb{R}$  such that  $K_h(\psi_h, \phi_h, \tilde{\mathbf{S}}_h) < C$  for all  $h > 0$  small enough. Therefore  $(\psi_h, \phi_h, \tilde{\mathbf{S}}_h) \in \mathcal{A}$ . There are now three things we are going to prove:

(i) the triple  $(\psi, \phi, \tilde{\mathbf{S}})$  satisfies

$$\psi \in W^{1,p}(\Omega^0; \mathbb{R}^3), \quad \psi|_{x_3=-1} = \text{id}, \quad \psi|_{x_3=0} = \phi|_{x_3=0}; \quad (4.3.12)$$

(ii) the admissible  $(\phi, \tilde{\mathbf{S}})$  are necessarily from  $\mathcal{B}_0$ ;

(iii) the desired inequality (4.3.9) holds.

Due to strong convergence of the sequence  $(\psi_h, \phi_h, \tilde{\mathbf{S}}_h)_{h>0}$  we know that  $\|\psi_h\|_{L^p(\Omega^0; \mathbb{R}^3)}$  and  $\|\phi_h\|_{L^q(\Omega^1; \mathbb{R}^3)}$  are uniformly bounded. According to (4.3.2) and (4.3.4) we have that  $\|\psi_h\|_{W^{1,p}(\Omega^0; \mathbb{R}^3)}$ ,  $\|\psi_h\|_{L^p(\Gamma^{-1}; \mathbb{R}^3)}$ ,  $\|\psi_h\|_{L^p(\Gamma^0; \mathbb{R}^3)}$  and  $\|\phi_h\|_{L^q(\Gamma^0; \mathbb{R}^3)}$  are uniformly bounded. The first consequence is that there is a subsequence of  $(\psi_h)_{h>0}$  that converges weakly in  $W^{1,p}(\Omega^0; \mathbb{R}^3)$ , so by the uniqueness of limits we have that  $\psi \in W^{1,p}(\Omega^0; \mathbb{R}^3)$ . Secondly,

there are weakly convergent subsequences (in  $L^p(\Gamma^{-1}; \mathbb{R}^3)$ ,  $L^p(\Gamma^0; \mathbb{R}^3)$  and  $L^q(\Gamma^0; \mathbb{R}^3)$ ) of sequences  $(\psi_h|_{x_3=-1})_{h>0}$ ,  $(\psi_h|_{x_3=0})_{h>0}$  and  $(\phi_h|_{x_3=0})_{h>0}$ . Since for all  $h > 0$  we have  $(\psi_h, \phi_h, \tilde{S}_h) \in \mathcal{A}$ , we can apply weak convergences in equations  $\psi_h|_{x_3=-1} = \text{id}$  and  $\psi_h|_{x_3=0} = \phi_h|_{x_3=0}$  to finally obtain (4.3.12), i.e. the part (i).

Since  $g$  is nonnegative, we have that  $K_h(\psi_h, \phi_h, \tilde{S}_h) \geq J_h(\phi_h, \tilde{S}_h)$ , so specially

$$\begin{aligned} +\infty > L &= \lim_{h \rightarrow 0} K_h(\psi_h, \phi_h, \tilde{S}_h) = \liminf_{h \rightarrow 0} K_h(\psi_h, \phi_h, \tilde{S}_h) \\ &\geq \liminf_{h \rightarrow 0} J_h(\phi_h, \tilde{S}_h) \geq J_0(\phi, \tilde{S}). \end{aligned}$$

Thus  $J_0(\phi, \tilde{S}) < +\infty$ , and consequently  $(\phi, \tilde{S}) \in \mathcal{B}_0$ , i.e. we have the part (ii). Finally, the part (iii) is a consequence of superadditivity of  $\liminf$ .

Let us now prove the second part of the proposition. Let us take any  $(\psi, \phi, \tilde{S}) \in \mathcal{A}_0$ . Let  $(\phi_h, \tilde{S}_h)_{h>0} \subset \mathcal{B}$  be a family converging strongly to  $(\phi, \tilde{S})$  in  $L^q \times \text{SO}(3)$  such that  $J_h(\phi_h, \tilde{S}_h) \rightarrow J_0(\phi, \tilde{S})$  and with condition (4.3.10) satisfied. Due to definition of quasiconvex envelope of  $G$ , there exists  $(\psi_h)_{h>0} \subset W^{1,p}(\Omega^0; \mathbb{R}^3)$  such that  $\psi_h \rightarrow \psi$  in  $L^p(\Omega^0; \mathbb{R}^3)$  and additionally  $G(\psi_h) \rightarrow G_0(\psi)$ . Due to (4.3.4),  $\|\psi_h\|_{W^{1,p}(\Omega^0; \mathbb{R}^3)}$  is uniformly bounded. Due to (4.3.10)  $\|\phi_h\|_{W^{1,q}(\Gamma^0; \mathbb{R}^3)}$  is uniformly bounded as well.

Let us apply Lemma 4.3.4 for  $(\psi_h)_{h>0}$  and  $(\phi_h|_{x_3=0})_{h>0}$ . We obtain  $(\bar{\psi}_h)_{h>0} \subset W^{1,p}(\Omega^0; \mathbb{R}^3)$  such that functions  $\bar{\psi}_h$  are small perturbations of  $\psi_h$  (in a sense that we still have properties  $\bar{\psi}_h \rightarrow \psi$  in  $L^p(\Omega^0; \mathbb{R}^3)$  and  $\limsup_{h \rightarrow 0} G(\bar{\psi}_h) \leq G_0(\psi)$ ), but with the property  $\bar{\psi}_h|_{x_3=0} = \phi_h|_{x_3=0}$ .

Since  $\bar{\psi}_h|_{x_3=0} = \phi_h$  for all  $h > 0$  and  $J_h(\phi_h, \tilde{S}_h) \rightarrow J_0(\phi, \tilde{S})$ , we finally obtain that for sequence  $(\bar{\psi}_h, \phi_h, \tilde{S}_h)_{h>0}$  we have desired property: it strongly converges to  $(\psi, \phi, \tilde{S})$  and satisfies

$$\limsup_{h \rightarrow 0} (G(\bar{\psi}_h) + J_h(\phi_h, \tilde{S}_h)) \leq G_0(\psi) + J_0(\phi, \tilde{S}). \quad (4.3.13)$$

□

**Lemma 4.3.4 ([8]).** Let  $1 < p \leq q < \infty$ . Let there be a functional  $G : L^p(\Omega^0; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  defined as

$$G(\psi) = \begin{cases} \int_{\Omega^0} g(\nabla \psi) dx & \psi \in W^{1,p}(\Omega^0; \mathbb{R}^3), \psi|_{x_3=-1} = \text{id}, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.3.14)$$



with  $g : \mathbb{M}^{3 \times 3} \rightarrow \mathbb{R}$  satisfying the growth condition  $0 \leq g(\mathbf{X}) \leq C(1 + \|\mathbf{X}\|_F^q)$  for all  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ . Let  $G_0(\psi) := \Gamma - G(\psi)$ . Let  $(\psi, \tilde{\phi}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{1,q}(\omega; \mathbb{R}^3)$  with  $\psi|_{x_3=0} = \tilde{\phi}$ . Let there be  $(\psi_n)_{n \geq 1} \subset W^{1,p}(\Omega^0; \mathbb{R}^3)$  and  $(\tilde{\phi}_n)_{n \geq 1} \subset W^{1,q}(\omega; \mathbb{R}^3)$  such that

$$\psi_n \rightarrow \psi \text{ in } L^p(\Omega^0; \mathbb{R}^3), \tilde{\phi}_n \rightarrow \tilde{\phi} \text{ in } L^q(\omega; \mathbb{R}^3), \lim_{n \rightarrow \infty} G(\psi_n) = G_0(\psi) \quad (4.3.15)$$

and such that  $(\|\nabla \psi_n\|_{L^p(\Omega^0; \mathbb{R}^{3 \times 3})})_{n \geq 1}$  and  $(\|\nabla' \tilde{\phi}_n\|_{L^q(\omega; \mathbb{R}^{3 \times 2})})_{n \geq 1}$  are uniformly bounded. Then there exists  $(\bar{\psi}_n)_{n \geq 1} \subset W^{1,p}(\Omega^0; \mathbb{R}^3)$  (up to a subsequence) such that

$$\bar{\psi}_n \rightarrow \psi \text{ in } L^p(\Omega^0; \mathbb{R}^3), \bar{\psi}_n|_{x_3=0} = \tilde{\phi}_n, \text{ and } \limsup_{n \rightarrow +\infty} G(\bar{\psi}_n) \leq G_0(\psi). \quad (4.3.16)$$

*Proof.* This claim is originally a part of Lemma 2 in [8]. For the completeness of the thesis, we hereby present its proof.

Note that since  $p \leq q$  we have that  $L^q(\omega; \mathbb{R}^3) \subset L^p(\omega; \mathbb{R}^3)$ , so all assumptions regarding  $L^q$  spaces for functions  $(\tilde{\phi}_n)_{n \geq 1}$  hold for  $L^p$  spaces as well.

The idea of the proof is to change the behaviour of the functions  $\psi_n$  near  $\Gamma^0$  such that the trace becomes equal to the functions  $\tilde{\phi}_n$ , keeping the value of  $G(\psi_n)$  controlled.

From the assumptions, we see that  $\tilde{\phi}, \tilde{\phi}_n \in W^{1,p}(\omega; \mathbb{R}^3)$ . This is why we can identify those functions with functions defined on  $\Omega^0$  ( $\phi_n(\mathbf{x}', x_3) := \tilde{\phi}_n(\mathbf{x}'), \phi(\mathbf{x}', x_3) := \tilde{\phi}(\mathbf{x}')$ ). Those functions are from  $W^{1,p}(\Omega^0; \mathbb{R}^3)$ , moreover, they are again uniformly bounded by the same constant since  $\partial_3 \phi = \partial_3 \phi_n = 0$ .

Since  $(\|\nabla' \tilde{\phi}_n\|_{L^p(\omega; \mathbb{R}^{3 \times 2})})_{n \geq 1}$  is uniformly bounded and since  $\partial_3 \psi \in L^p(\Omega^0; \mathbb{R}^3)$ , for every  $m \in \mathbb{N}$  there exists  $\eta > 0$  such that for  $\Sigma_\eta := \omega \times \langle -\eta, 0 \rangle$

$$\begin{aligned} \int_{\Sigma_\eta} (1 + \|\nabla \phi_n\|_F^p) &\leq \frac{1}{m}, \\ \int_{\Sigma_\eta} \|\partial_3 \psi\|^p &\leq \frac{1}{m^p}. \end{aligned} \quad (4.3.17)$$

We use slicing method of De Giorgi. For  $i = 1, \dots, m$  we define  $\Sigma_i := \omega \times \langle -i\frac{\eta}{m}, 0 \rangle$  and decreasing sequence  $U_i := \Omega^0 \setminus \Sigma_i$  of subsets of  $\Omega^0$ .

Let  $(\varphi_i)_{i=1, \dots, m-1}$  be a sequence of functions in  $C^1(\mathbb{R}^3; \mathbb{R})$  satisfying  $\varphi_i(\mathbb{R}^3) \in [0, 1]$ ,  $\varphi_i = 1$  on  $U_{i+1}$ ,  $\varphi_i = 0$  on  $\Sigma_i$ ,  $\|\nabla \varphi\|_{L^p(\Omega^0; \mathbb{R}^3)} \leq C_1 \frac{m}{\eta}$ . We now define  $\bar{\psi}_{n,i} := \varphi_i(\psi_n - \phi_n) + \phi_n$  on  $\Omega^0$ . Since  $\psi_{n,i}$  near  $\omega$  behaves as  $\phi_n$ , we have  $\psi_{n,i}|_{x_3=0} = \tilde{\phi}_n$ . Let us see

the error made: for each  $i = 0, \dots, m-1$  we have

$$\begin{aligned} G(\bar{\psi}_{n,i}) &= \int_{\Omega^0} g(\nabla \bar{\psi}_{n,i}) d\mathbf{x} \\ &= \int_{\Sigma_i} g(\nabla \bar{\psi}_{n,i}) d\mathbf{x} + \int_{\Sigma_{i+1} \setminus \Sigma_i} g(\nabla \bar{\psi}_{n,i}) d\mathbf{x} + \int_{U_{i+1}} g(\nabla \bar{\psi}_{n,i}) d\mathbf{x} \\ &\leq \int_{\Sigma_\eta} g(\nabla \phi_n) d\mathbf{x} + \int_{\Sigma_{i+1} \setminus \Sigma_i} g(\nabla \bar{\psi}_{n,i}) d\mathbf{x} + \int_{\Omega^0} g(\nabla \psi_n) d\mathbf{x}. \end{aligned} \quad (4.3.18)$$

We now bound terms on the right hand side. In the first term, we use growth condition on  $g$  and (4.3.17) to obtain

$$\int_{\Sigma_\eta} g(\nabla \phi_n) d\mathbf{x} \leq C \int_{\Sigma_\eta} (1 + \|\nabla \phi_n\|_F^p) d\mathbf{x} \leq \frac{C}{m}. \quad (4.3.19)$$

For the second term on the right hand side of (4.3.18) we will again use the growth condition. Note that since  $\bar{\psi}_{n,i} = (1 - \varphi_i)(\phi_n - \psi_n) + \psi_n$  we have

$$\begin{aligned} \|\nabla \bar{\psi}_{n,i}\|_F &\leq C_1 \frac{m}{\eta} \|\phi_n - \psi_n\| + \|\nabla \phi_n - \nabla \psi_n\|_F + \|\nabla \psi_n\|_F \\ &\leq C_1 \frac{m}{\eta} \|\phi_n - \psi_n\| + 2\|\nabla \psi_n\|_F + \|\nabla \phi_n\|_F. \end{aligned} \quad (4.3.20)$$

Applying obtained to (4.3.18) we get

$$\begin{aligned} G(\bar{\psi}_{n,i}) &\leq C' \left( \frac{1}{m} + \int_{\Sigma_{i+1} \setminus \Sigma_i} \left( \frac{m}{\eta} \right)^p \|\phi_n - \psi_n\|^p + \|\nabla \psi_n\|_F^p \right. \\ &\quad \left. + \|\nabla \phi_n\|_F^p d\mathbf{x} \right) + \int_{\Omega^0} g(\nabla \psi_n) d\mathbf{x}, \end{aligned} \quad (4.3.21)$$

for  $i = 0, \dots, m$ . By averaging these  $m$  inequalities, we obtain

$$\begin{aligned} \frac{1}{m} \sum_{i=0}^{m-1} G(\bar{\psi}_{n,i}) &\leq C' \left( \frac{1}{m} + \frac{1}{m} \int_{\Sigma_\eta} \left( \frac{m}{\eta} \right)^p \|\phi_n - \psi_n\|^p + \|\nabla \psi_n\|_F^p + \|\nabla \phi_n\|_F^p d\mathbf{x} \right) \\ &\quad + \int_{\Omega^0} g(\nabla \psi_n) d\mathbf{x}. \end{aligned} \quad (4.3.22)$$

Since  $\|\nabla \psi_n\|_{L^p(\Omega^0; \mathbb{R}^{3 \times 3})}$  and  $\|\nabla \phi_n\|_{L^p(\Omega^0; \mathbb{R}^{3 \times 3})} = \|\nabla' \tilde{\phi}_n\|_{L^p(\omega; \mathbb{R}^{3 \times 2})}$  are uniformly bounded from above, we can write

$$\frac{1}{m} \sum_{i=0}^{m-1} G(\bar{\psi}_{n,i}) \leq C'' \left( \frac{1}{m} + \frac{1}{m} \int_{\Sigma_\eta} \left( \frac{m}{\eta} \right)^p \|\phi_n - \psi_n\|^p d\mathbf{x} \right) + \int_{\Omega^0} g(\nabla \psi_n) d\mathbf{x}. \quad (4.3.23)$$

Let  $i(n, m)$  be the index such that

$$G(\bar{\psi}_{n,i(n,m)}) = \min_{i=0, \dots, m-1} G(\bar{\psi}_{n,i}). \quad (4.3.24)$$

We know that this term is less or equal to the average over all  $i$ , so

$$G(\bar{\psi}_{n,i(n,m)}) \leq C'' \left( \frac{1}{m} + \frac{1}{m} \int_{\Sigma_\eta} \left( \frac{m}{\eta} \right)^p \|\phi_n - \psi_n\|^p d\mathbf{x} \right) + \int_{\Omega^0} g(\nabla \psi_n) d\mathbf{x}. \quad (4.3.25)$$

Taking the lim sup with respect to  $n \in \mathbb{N}$  on the both sides, we obtain

$$\limsup_{n \rightarrow \infty} G(\bar{\psi}_{n,i(n,m)}) \leq C'' \left( \frac{1}{m} + \frac{1}{m} \int_{\Sigma_\eta} \left( \frac{m}{\eta} \right)^p \|\phi - \psi\|^p d\mathbf{x} \right) + G_0(\psi). \quad (4.3.26)$$

Using Lemma 4.3.5 for  $\mathbf{u} = \phi - \psi$ ,  $\partial_3 \phi = 0$  and (4.3.17), we obtain

$$\int_{\Sigma_\eta} \|\phi - \psi\|^p d\mathbf{x} \leq C\eta^p \int_{\Sigma_\eta} \|\partial_3 \psi\|^p d\mathbf{x} \leq C_P \left( \frac{\eta}{m} \right)^p. \quad (4.3.27)$$

Thus we obtain

$$\limsup_{n \rightarrow \infty} G(\bar{\psi}_{n,i(n,m)}) \leq \frac{C'''}{m} + G_0(\psi). \quad (4.3.28)$$

By taking lim sup with respect to  $m \in \mathbb{N}$  and by the diagonalization argument, we conclude that there exists mapping  $n \rightarrow m(n)$  such that

$$\limsup_{n \rightarrow \infty} G(\bar{\psi}_{n,i(n,m(n))}) \leq G_0(\psi). \quad (4.3.29)$$

We define  $\bar{\psi}_n := \bar{\psi}_{n,i(n,m(n))}$ . Let us prove that  $\bar{\psi}_n \rightarrow \psi$  strongly in  $L^p(\Omega^0; \mathbb{R}^3)$ :

$$\begin{aligned} \int_{\Omega^0} \|\bar{\psi}_n - \psi\|^p d\mathbf{x} &= \int_{\Omega^0} \|(1 - \varphi_{i(n,m(n))})(\phi_n - \psi_n) + (\psi_n - \psi)\|^p d\mathbf{x} \\ &\leq C \left( \int_{\Sigma_\eta(n)} \|\phi_n - \psi_n\|^p d\mathbf{x} + \int_{\Omega^0} \|\psi_n - \psi\|^p d\mathbf{x} \right) \\ &\leq C \left( \int_{\Sigma_\eta(n)} \|\phi_n - \phi\|^p d\mathbf{x} + \int_{\Sigma_\eta(n)} \|\phi - \psi\|^p d\mathbf{x} + 2 \int_{\Omega^0} \|\psi - \psi_n\|^p d\mathbf{x} \right). \end{aligned} \quad (4.3.30)$$

For the second term on the right hand side we use (4.3.27), in other two terms the strong convergence of  $(\phi_n)_{n \geq 1}$  and  $(\psi_n)_{n \geq 1}$ , and thus the strong convergence of  $(\bar{\psi}_n)_{n \geq 1}$  is proved.  $\square$

**Lemma 4.3.5.** Let there be a bounded Lipschitz set  $\omega$ . Let for any  $\eta$  define the set  $\Sigma_\eta := \omega \times [-\eta, 0]$ . There exists a constant  $C_P > 0$  such that the Poincaré–type estimate

$$\|\mathbf{u}\|_{L^p(\Sigma_\eta; \mathbb{R}^3)}^p \leq C_P \eta^p \|\partial_3 \mathbf{u}\|_{L^p(\Sigma_\eta; \mathbb{R}^3)}^p \quad (4.3.31)$$

holds for any  $\eta > 0$  and any  $\mathbf{u} \in W^{1,p}(\Sigma_\eta; \mathbb{R}^3)$  with  $\mathbf{u}|_{x_3=0} = 0$ .

*Proof.* Let us now take any nonnegative  $\xi \in W^{1,1}(\Sigma_\eta; \mathbb{R})$ . For a.e.  $\mathbf{x} = (\mathbf{x}', x_3) \in \Sigma_\eta$  we have

$$\xi(\mathbf{x}', x_3) \leq \left| \int_{x_3}^0 \partial_3 \xi(\mathbf{x}', y_3) dy_3 \right| \leq \int_{-\eta}^0 |\partial_3 \xi(\mathbf{x}', y_3)| dy_3. \quad (4.3.32)$$

After integrating over  $\mathbf{x}' \in \omega$  and then  $x_3 \in [-\eta, 0]$  we obtain

$$\int_{\Sigma_\eta} \xi d\mathbf{x}' \leq \eta \int_{\Sigma_\eta} |\partial_3 \xi| d\mathbf{x}. \quad (4.3.33)$$

We will use this inequality for  $\xi(\mathbf{x}) = |\mathbf{u}(\mathbf{x})|^p$ :

$$|\partial_3 \xi(\mathbf{x})| \leq p |\mathbf{u}|^{p-1} |\partial_3 \mathbf{u}|. \quad (4.3.34)$$

We use Hölder inequality to obtain

$$\|\mathbf{u}\|_{L^p(\Sigma_\eta; \mathbb{R}^3)}^p \leq p\eta \int_{\Sigma_\eta} |\mathbf{u}|^{p-1} |\partial_3 \mathbf{u}| d\mathbf{x} \leq p\eta \|\mathbf{u}\|_{L^p(\Sigma_\eta; \mathbb{R}^3)}^{p-1} \|\partial_3 \mathbf{u}\|_{L^p(\Sigma_\eta; \mathbb{R}^3)}. \quad (4.3.35)$$

We obtain the inequality (4.3.31) after dividing both sides of the last inequality with  $\|\mathbf{u}\|_{L^p(\Sigma_\eta; \mathbb{R}^3)}^{p-1}$  and lifting them to the power of  $p$ .  $\square$

We finish with two more technical results.

**Lemma 4.3.6.** There are constants  $c > 0$ ,  $C \in \mathbb{R}$  such that the inequality

$$\begin{aligned} \|\psi\|_{L^p(\Omega^0; \mathbb{R}^3)}^p + \int_{\Omega^0} W^0(\nabla \psi) d\mathbf{x} \\ \geq c \left( \|\psi\|_{W^{1,p}(\Omega^0; \mathbb{R}^3)}^p + \|\psi\|_{L^p(\Gamma^{-1}; \mathbb{R}^3)}^p + \|\psi\|_{L^p(\Gamma^0; \mathbb{R}^3)}^p \right) - C \end{aligned} \quad (4.3.36)$$

holds for all  $\psi \in W^{1,p}(\Omega^0; \mathbb{R}^3)$ .

*Proof.* From coercivity of  $W^0$  (condition W0.3)), for  $c'_1 = \min\{1, c_1\}$  we obtain

$$\|\psi\|_{L^p(\Omega^0; \mathbb{R}^3)}^p + \int_{\Omega^0} W^0(\nabla \psi) d\mathbf{x} \geq c'_1 \|\psi\|_{W^{1,p}(\Omega^0; \mathbb{R}^3)}^p - c_2. \quad (4.3.37)$$

Two other terms on the right hand side of (4.3.36) we obtain by applying the trace theorem on the term  $\|\psi\|_{W^{1,p}(\Omega^0; \mathbb{R}^3)}$ , with editing constants  $c'_1, c_2$ .  $\square$

**Lemma 4.3.7.** Let  $n \geq 1$  and  $s \in [0, 1]$ . Let  $(\phi_h)_{h>0}$  be uniformly bounded family in  $W^{1,2}(\Omega^1; \mathbb{R}^3)$ , with

$$\phi_h = \sum_{k=0}^n h^k (x_3 - s)^k \tilde{\phi}_h^k, \quad (\tilde{\phi}_h^k)_{h>0} \subset W^{1,2}(\omega; \mathbb{R}^3), \quad k \in \{0, \dots, n\}. \quad (4.3.38)$$

Then  $(\phi_h|_{x_3=0})_{h>0}$  is uniformly bounded in  $W^{1,2}(\omega; \mathbb{R}^3)$ .

*Proof.* Without loss of generality, let  $s = 0$ . If this is not the case, we will perform the proof for functions  $(\tilde{\psi}_h^k)_{h>0} \subset W^{1,2}(\omega; \mathbb{R}^3)$  for which

$$\sum_{k=0}^n h^k (x_3 - s)^k \tilde{\phi}_h^k = \sum_{k=0}^n h^k x_3^k \tilde{\psi}_h^k.$$

Note that now  $\phi_h|_{x_3=0} = \tilde{\phi}_h^0$ , for all  $h > 0$ .

For  $\beta \in \{1, 2\}$ , we calculate

$$\begin{aligned} C &\geq \|\partial_\beta \phi_h\|_{L^2(\Omega^1; \mathbb{R}^3)}^2 = \int_\omega \int_0^1 \left( \sum_{k=0}^n h^k x_3^k \partial_\beta \tilde{\phi}_h^k \right)^2 d\mathbf{x} \\ &= \int_\omega \int_0^1 \sum_{k_1=0}^n \sum_{k_2=0}^n x_3^{k_1+k_2} h^{k_1+k_2} \partial_\beta \tilde{\phi}_h^{k_1} \cdot \partial_\beta \tilde{\phi}_h^{k_2} d\mathbf{x} \\ &= \int_\omega \sum_{k_1=0}^n \sum_{k_2=0}^n \frac{1}{k_1 + k_2 + 1} h^{k_1+k_2} \partial_\beta \tilde{\phi}_h^{k_1} \cdot \partial_\beta \tilde{\phi}_h^{k_2} d\mathbf{x}' \quad (4.3.39) \\ &\geq \lambda_{\min}(\mathbf{H}_{n+1}) \int_\omega \sum_{k=0}^n \left\| h^k \partial_\beta \tilde{\phi}_h^k \right\|^2 d\mathbf{x}' \\ &\geq \lambda_{\min}(\mathbf{H}_{n+1}) \int_\omega \left\| \partial_\beta \tilde{\phi}_h^0 \right\|^2 d\mathbf{x}' \\ &= \lambda_{\min}(\mathbf{H}_{n+1}) \|\partial_\beta \tilde{\phi}_h^0\|_{L^2(\omega; \mathbb{R}^3)}^2, \end{aligned}$$

where  $\lambda_{\min}(\mathbf{H}_N) > 0$  is the smallest eigenvalue of the  $N \times N$  Hilbert matrix (the inequality is the consequence of the fact that the Hilbert matrix is positive definite). For  $\|\tilde{\phi}_h\|_{L^2(\omega; \mathbb{R}^3)}^2$  we perform analogous calculation.  $\square$

## 4.4. PROOF OF THEOREM 4.2.1

*Proof of Theorem 4.2.1.* We prove the theorem by using Proposition 4.3.1 for each part of the theorem separately.

Let us take any  $\alpha \in \langle -1, 5/3 \rangle \cup [2, +\infty)$ . Let us define family of functionals  $(J_h)_{h>0}$  by  $J_h(\phi) = J_h^{3d,\alpha}(\phi)$  (defined in (3.1.2)), for all  $\phi \in L^q(\Omega^1; \mathbb{R}^3)$  (in the view of Remark 4.3.3, each functional  $J_h$  is independent of matrices  $\tilde{S} \in L^2(\omega; \text{SO}(3))$ ), and its set of admissible functions is  $W^{1,p}(\Omega^1; \mathbb{R}^3)$ . According to Theorem 3.2.1, the  $\Gamma$ -limit of the family  $(J_h)_{h>0}$  is  $J_0 = J_0^{3d,\alpha}$ , depending on the value of  $\alpha$ . Due to Lemma 3.3.1, family  $(J_h)_{h>0}$  satisfies the condition (4.3.2).

Let us define  $g(\mathbf{X}) = W^0(\mathbf{X})$ , for all  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ . Then we have  $G(\psi) = \mathcal{W}^0(\psi)$ , where  $\mathcal{W}^0$  is defined in (4.1.3), and

$$G_0(\psi) = \Gamma - \mathcal{W}^0(\psi) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} & \psi \in W^{1,p}(\Omega^0; \mathbb{R}^3), \psi|_{x_3=-1} = \text{id}, \\ +\infty & \text{otherwise.} \end{cases} \quad (4.4.1)$$

Due to assumptions W0.2) and W0.4)  $g$  satisfies the nonnegativity conditions and the growth condition. Due to Lemma 4.3.6  $G$  satisfies the condition (4.3.4).

Since for all  $\alpha$  and  $h > 0$  the functional  $K_h^{3d,\alpha}$  from (4.1.4) can be written as

$$K_h^{3d,\alpha}(\psi, \tilde{\phi}) = \begin{cases} \mathcal{W}^0(\psi) + J_h^{3d,\alpha}(\phi) & \psi|_{x_3=0} = \phi|_{x_3=0}, \psi|_{x_3=-1} = \text{id}, \\ +\infty & \text{otherwise,} \end{cases}$$

which is of the form (4.3.5), we conclude that we have  $K_h = K_h^{3d,\alpha}$  for all  $h > 0$  and  $\mathcal{A} = \mathcal{A}^{3d,\alpha}$ . Also, functionals  $K_0^{3d,\alpha}$  from all parts of Theorem 4.2.1 are given by

$$K_0^{3d,\alpha}(\psi, \tilde{\phi}) = \begin{cases} \Gamma - \mathcal{W}^0(\psi) + J_0^{3d,\alpha}(\phi) & \psi|_{x_3=0} = \phi|_{x_3=0}, \psi|_{x_3=-1} = 0, \phi \in \mathcal{B}_0^{3d,\alpha}, \\ +\infty & \text{otherwise,} \end{cases}$$

(with  $J_0^{3d,\alpha}$  and  $\mathcal{B}_0^{3d,\alpha}$  are from particular parts of Theorem 3.2.1). This is of the form as in (4.3.7), so we conclude that  $K_0 = K_0^{3d,\alpha}$ .

Since all assumptions of Proposition 4.3.1.a) on  $(K_h)_{h>0}$  are fulfilled, we can apply it and conclude that for each for all  $(\psi, \phi) \in L^p(\Omega^0; \mathbb{R}^3) \times L^q(\Omega^1; \mathbb{R}^3)$  and for all

$(\psi_h, \phi_h)_{h>0} \subset L^p(\Omega^0; \mathbb{R}^3) \times L^q(\Omega^1; \mathbb{R}^3)$  converging strongly to  $(\psi, \phi)$  in  $L^p(\Omega^0; \mathbb{R}^3) \times L^q(\Omega^1; \mathbb{R}^3)$  we have

$$\liminf_{h \rightarrow 0} K_h(\psi_h, \phi_h) \geq K_0(\psi, \phi). \quad (4.4.2)$$

By taking into account that  $K_h = K_h^{3d,\alpha}$  and  $K_0 = K_0^{3d,\alpha}$ , we conclude that each of five parts of Theorem 4.2.1 the particular functional  $K_0^{3d,\alpha}$  satisfies the "lim inf" part of the definition of the  $\Gamma$ -convergence for the family  $(K_h^{3d,\alpha})_{h>0}$ .

For the "lim sup" part we now have to check if for arbitrary  $\phi \in \mathcal{B}_0^{3d,\alpha}$  there exists  $(\phi_h)_{h>0} \subset \mathcal{B}^{3d,\alpha}$  converging strongly to  $\phi$  in  $L^q(\Omega^1; \mathbb{R}^3)$  such that  $J_h^{3d,\alpha}(\phi_h) \rightarrow J_0^{3d,\alpha}(\phi)$  and that the additional condition (4.3.10) is satisfied. For that we are going to use Remark 3.3.3, and then apply the part b) of Proposition 4.3.1.

Since from for  $\alpha \in \langle -1, 0 \rangle$  admissible functions  $\tilde{\phi}$  are in  $L^q(\omega; \mathbb{R}^3)$ , there is no guarantee that a strongly convergent sequence  $(\phi_n)_{n \geq 1} \subset W^{1,q}(\Omega^1; \mathbb{R}^3)$  (converging to  $\phi$  in  $L^q(\Omega^1; \mathbb{R}^3)$ ) satisfies the condition (4.3.10). Thus the part a) of Theorem 4.2.1 we will have to prove in a different way, which will be done at the end of this proof.

For  $\alpha = 0$ , from Remark 3.3.3.b) we see that the family  $(\phi_h)_{h>0} \subset W^{1,q}(\Omega^1; \mathbb{R}^3)$  in the "lim sup" part of the proof is of the form  $\phi_h = \tilde{\phi}^0 + h(x_3 - 1/2)\tilde{\phi}^1$  with some  $\tilde{\phi}^0, \tilde{\phi}^1 \in W^{1,q}(\omega; \mathbb{R}^3)$ ,  $h > 0$ . For that it holds

$$\|\phi_h\|_{W^{1,q}(\Gamma^0; \mathbb{R}^3)}^q \leq q \left( \|\tilde{\phi}^0\|_{W^{1,q}(\omega; \mathbb{R}^3)}^q + \frac{1}{2q} \|\tilde{\phi}^1\|_{W^{1,q}(\omega; \mathbb{R}^3)}^q \right)$$

for all  $h \in \langle 0, 1 \rangle$ , thus the condition (4.3.10) holds, so the b) part of Proposition 4.3.1 holds, and consequently Theorem 4.2.1.b) is proved.

For  $\alpha \in \langle 2, +\infty \rangle$  from Remark 3.3.3.e) we see that the family of functions  $(\phi_h)_{h>0} \subset W^{1,q}(\Omega^1; \mathbb{R}^3)$  in the "lim sup" part of the proof is of the form  $\phi_h = \tilde{\phi}^0 + hx_3h\tilde{\phi}^1$  with some  $\tilde{\phi}^0, \tilde{\phi}^1 \in W^{1,q}(\omega; \mathbb{R}^3)$ ,  $h > 0$ . Then, similarly as for proof for Theorem 4.2.1.b), proof of Theorem 4.2.1.e) follows.

For cases  $\alpha \in \langle 0, 5/3 \rangle$  and  $\alpha = 2$  from Remark 3.3.3.c) and Remark 3.3.3.d), we see that the "lim sup" part of the proof for arbitrary admissible  $\tilde{\phi} \in L^2(\omega; \mathbb{R}^3)$  is obtained by families  $(\phi_h)_{h>0} \subset W^{1,2}(\Omega^1; \mathbb{R}^3)$  of the form

$$\phi_h = \sum_{k=0}^n h^k (x_3 - s)^k \tilde{\phi}_h^k, \quad k \in \{0, \dots, n\}. \quad (4.4.3)$$

for particular  $n \geq 1$  and  $s \in [0, 1]$ . Since this family  $(\phi_h)_{h>0}$  converges to  $\phi$  in  $L^2(\Omega^1; \mathbb{R}^3)$ , it is uniformly bounded in the same topology. By Lemma 3.3.1  $(\phi_h)_{h>0}$

is uniformly bounded in  $W^{1,2}(\Omega^0; \mathbb{R}^3)$ , so by Lemma 4.3.7  $(\phi_h|_{x_3=0})_{h>0}$  is uniformly bounded in  $W^{1,2}(\omega; \mathbb{R}^3)$ . Thus, condition (4.3.10) is satisfied as well, and this finishes the proof of Theorem 4.2.1.c) and Theorem 4.2.1.d).

The only thing left to prove is Theorem 4.2.1.a), more precisely, the "lim sup" part of the definition of the  $\Gamma$ –limit for the family  $(K_h^{3d, \langle -1, 0 \rangle})_{h>0}$ . Let us take arbitrary  $(\psi, \tilde{\phi}) \in \mathcal{A}_0^{3d, \langle -1, 0 \rangle}$ . We will prove that for all  $\varepsilon > 0$  there exists  $h^* > 0$  and  $(\psi^*, \tilde{\phi}^*) \in \mathcal{A}^{3d, \langle -1, 0 \rangle}$  such that

$$\begin{aligned} \|\psi^* - \psi\|_{L^p(\Omega^0; \mathbb{R}^3)} &\leq \varepsilon, \quad \|\tilde{\phi}^* - \tilde{\phi}\|_{L^q(\omega; \mathbb{R}^3)} \leq \varepsilon \\ \text{and } K_{h^*}^{3d, \langle -1, 0 \rangle}(\psi^*, \phi^*) - K_0^{3d, \langle -1, 0 \rangle}(\psi, \tilde{\phi}) &\leq \varepsilon. \end{aligned} \quad (4.4.4)$$

Note that without loss of generality assume that  $\mathbf{f} = 0$ , since the source term is linear and does not affect the convergence. Now we have that  $J_0^{3d, \langle -1, 0 \rangle}(\tilde{\phi}) = 0$  for all  $\tilde{\phi} \in L^q(\omega; \mathbb{R}^3)$ .

Let us take arbitrary  $\varepsilon > 0$ . Firstly, let  $(\psi_m)_{m \geq 1} \subset C^\infty(\omega; \mathbb{R}^3)$  with  $\psi_m|_{x_3=-1} = \text{id}$  be smooth  $W^{1,p}(\Omega^0; \mathbb{R}^3)$  approximations of  $\psi$ . Since  $W^0$  satisfies the growth condition W0.4) (and so does  $QW^0$ ), due to the Lebesgue dominated convergence theorem (dominated by  $\mathbf{x} \mapsto C(1 + \|\nabla \psi(\mathbf{x})\|_F^p)$ ) we get

$$\Gamma - \mathcal{W}^0(\psi_m) = \int_{\Omega^0} QW^0(\nabla \psi_m) d\mathbf{x} \rightarrow \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} = \Gamma - \mathcal{W}^0(\psi). \quad (4.4.5)$$

So there is  $m \in \mathbb{N}$  and  $\psi_\infty := \psi_m$  such that

$$\|\psi_\infty - \psi\|_{W^{1,p}(\Omega^0; \mathbb{R}^3)} < \frac{\varepsilon}{2} \text{ and } |(\Gamma - \mathcal{W}^0(\psi_\infty)) - (\Gamma - \mathcal{W}^0(\psi))| < \frac{\varepsilon}{3}. \quad (4.4.6)$$

Due to the trace theorem, we additionally demand that

$$\|\tilde{\phi}_\infty - \tilde{\phi}\|_{L^q(\omega; \mathbb{R}^3)} = \|\psi_\infty - \psi\|_{L^q(\Gamma^0; \mathbb{R}^3)} < \varepsilon, \quad (4.4.7)$$

where  $\tilde{\phi}_\infty := \psi_\infty|_{x_3=0}$ .

Let us define a constant sequence  $\tilde{\phi}_n = \tilde{\phi}_\infty$ . Due to Remark 3.3.3.a), for this sequence we have  $J_{1/n}^{3d, \langle -1, 0 \rangle}(\phi_n) \rightarrow J_0^{3d, \langle -1, 0 \rangle}(\tilde{\phi}_\infty) = 0$ . Due to the definition of weakly lower semicontinuous envelope, there exists a sequence  $(\psi_n)_{n \geq 1}$  such that  $\psi_n \rightharpoonup \psi_\infty$  in  $W^{1,p}(\Omega^0; \mathbb{R}^3)$  and  $\mathcal{W}^0(\psi_n) d\mathbf{x} \rightarrow \Gamma - \mathcal{W}^0(\psi_\infty)$ . Let us apply Lemma 4.3.4 for sequence



$\psi_n \rightarrow \psi_\infty$  and the constant sequence  $\tilde{\phi}_n \rightarrow \psi_\infty|_{x_3=0}$ . We conclude that there exists  $(\bar{\psi}_n)_{n \geq 1} \subset W^{1,p}(\Omega^0; \mathbb{R}^3)$  such that

$$\bar{\psi}_n \rightarrow \psi_\infty \text{ in } L^p(\Omega^0; \mathbb{R}^3), \bar{\psi}_n|_{x_3=0} = \phi_n|_{x_3=0} \text{ and } \limsup_{n \rightarrow +\infty} \mathcal{W}^0(\bar{\psi}_n) \leq \Gamma - \mathcal{W}^0(\psi_\infty). \quad (4.4.8)$$

We can conclude that there is  $n \in \mathbb{N}$  large enough such that:

$$\begin{aligned} \|\bar{\psi}_n - \psi_\infty\|_{L^p(\Omega^0; \mathbb{R}^3)} &< \frac{\varepsilon}{2} \text{ (due to (4.4.8))}, \\ \|\tilde{\phi}_n - \tilde{\phi}_\infty\|_{L^q(\omega; \mathbb{R}^3)} &= 0 \text{ (due to the definition of the sequence } (\tilde{\phi}_n)_{n \geq 1}), \\ J_{1/n}^{3d, \langle -1, 0 \rangle}(\phi_n) &\leq \frac{\varepsilon}{3} \text{ (due to Remark 3.3.3.a) and } J_0^{3d, \langle -1, 0 \rangle} \equiv 0), \\ \mathcal{W}^0(\bar{\psi}_n) - (\Gamma - \mathcal{W}^0(\psi_\infty)) &< \frac{\varepsilon}{3} \text{ (due to (4.4.8)).} \end{aligned} \quad (4.4.9)$$

Together with (4.4.6) and (4.4.7), from triangle inequalities we conclude that  $h^* = 1/n$  and  $(\psi^*, \tilde{\phi}^*) := (\bar{\psi}_n, \tilde{\phi}_n)$  satisfy (4.4.4) and conclude the proof for Theorem 4.2.1.a), and the proof for the whole Theorem 4.2.1.  $\square$

# 5. A NAGHDI TYPE NONLINEAR SHELL MODEL

## 5.1. INTRODUCTION

After the asymptotic analysis of the 3d–thin 3d model in various regimes and its interaction with an other (not thin) 3d body, now we proceed to proposition of adequate 3d–2d model that has the same convergence properties as the model observed in the Chapter 4. In order to do this, in this chapter we formulate a new nonlinear shell model that will be applicable in all situations, irrespective of the geometry, boundary conditions or scaling order of energy. This original model is already presented in [61], together with majority of properties given in this chapter.

The model is formulated in terms of two unknown functions  $\psi$ ,  $S$ , where  $\psi$  parametrizes the middle surface of the deformed shell and  $S$  is a function with values in rotations that describe the rotation of the cross-section of the shell. The cross-sections are allowed to shear with respect to the deformed middle surface, which is typical to the Naghdi type models. This is continuation of the research in the linear case and the formulation of a two-dimensional linear shell model of Naghdi type from [16, 87] (presented in Chapter 1), and the previous work for Koiter type models from [5, 14, 15, 86] and flexural shell model from [85].

The main features of the model we formulate are the following:

- The model is well defined for shells with undeformed geometry parametrized by  $W^{1,\infty}$  function. Approach of adding (infinitesimal) rotations in the model as ad-

ditional unknowns in order to extend the model for less regular geometries is well known, see e.g. [5, 16]. Similar considerations in the case of nonlinear rods can be found in [48].

- The model is frame indifferent.
- There are two terms in the elastic energy of the shell, one scaled by  $h$  and the other by  $h^3$ , where  $h$  is the thickness of the shell. This is also typical for the Koiter's and Naghdi's type models, see [5, 25, 26, 50]
- In the energy of the model all types of shell deformations can be recognized: membrane, shear, drill, flexural. Membrane, shear and flexural deformations are well known in shell theories, but inclusion of drilling rotations is also not new in the literature, see [12, 44, 45] for example. Thus the model can be considered as a 6-parameter shell model.
- Additional restriction of the unknowns that implies unshearability (cross-sections remain perpendicular to the deformed middle surface) and no twist of the cross-section (no drill) leads to a model that is a perturbation of the nonlinear Koiter model from [21, 47].
- Further restriction that rotation  $\mathbf{S}$  maps covariant vectors of undeformed shell to covariant vectors of deformed shell leads to the classical nonlinear flexural shell model from [40].
- Linearization of the strains in the proposed model leads to exactly the same strains as in the linear Naghdi's type model from [87], moreover if in addition we consider the model for the St. Venant–Kirchhoff material linearization leads exactly to the model from [87].
- Differential formulation of the model implies that it can be interpreted as a Cosserat model with one director, see [6].
- When the total energy functional is scaled by  $h$  it  $\Gamma$ -converges in the appropriate topology to the model with no flexural, no shear and no drill energy which cor-

responds to the nonlinear membrane shell model derived from three-dimensional nonlinear elasticity in [51].

- When the total energy functional is scaled by  $h^3$  it  $\Gamma$ -converges in the appropriate topology to the nonlinear flexural shell model derived from three-dimensional nonlinear elasticity in [40].
- In the case when the middle surface of undeformed geometry is planar and when the total energy functional is scaled by  $h^{\alpha+1}$ ,  $\alpha \in \langle 0, 5/3 \rangle$  it  $\Gamma$ -converges in the appropriate topology to the model with no energy contribution, with only conditions on the set of admissible functions being short maps, which corresponds to the nonlinear constrained membrane plate model derived from three-dimensional nonlinear elasticity in [30].
- When the total energy functional is scaled by  $h^{\alpha+1}$ ,  $\alpha \in \langle -1, 0 \rangle$ , in  $\Gamma$ -limit we obtain a model with no energy contribution and with no conditions in the set of admissible functions, meaning that in that regime the shell is so flexible so that appears as the shell is not there.
- When the total energy functional is scaled by  $h^{\alpha+1}$ ,  $\alpha \in \langle 2, +\infty \rangle$ , in  $\Gamma$ -limit we obtain a model with no energy contribution and with conditions on the set of admissible functions being only rigid transformations, meaning that in the limit the shell is so stiff that only rotations and translations of the shell are possible.

The last five items present the main mathematical contribution of this chapter through the  $\Gamma$ -convergence arguments.

In Section 5.2 we formulate the nonlinear shell model of Naghdi's type and sets of admissible deformations also for the Koiter's type model and the flexural model. In Section 5.3 we give relations between these sets of admissible deformations and analyse the models when considered on these subsets of deformations. We also linearise the model and obtain exactly the model from [87]. At the end of Section 5.3 we also derive the weak and differential formulation of the original formulation and conclude that the model can be recognized as the Cosserat model with one director for a particular constitutive law. In

Section 5.4 we do the asymptotic analysis with respect to the thickness  $h$  of the shell, using  $\Gamma$ -convergence, of the proposed model in five regimes, obtaining in the limit models similar to the ones presented in Chapter 3 (in the case of the planar undeformed geometry of the middle surface) and appearing in the literature. This is presented in Theorem 5.4.1 and Theorem 5.4.2. At the end of this section several technical lemmas for matrices and matrix functions are proved.

## 5.2. DEFINITION OF THE MODEL

Let  $\omega \subset \mathbb{R}^2$  be an open bounded and simply connected set with a Lipschitz-continuous boundary and let  $\theta : \bar{\omega} \rightarrow \mathbb{R}^3$  be an injective mapping that belongs to  $W^{1,\infty}(\omega; \mathbb{R}^3)$ . We use notations from Chapter 1.2 for (a.e. linearly independent) vectors  $\mathbf{a}_\beta(\mathbf{x}') = \partial_\beta \theta(\mathbf{x}')$ , 2-surface  $S = \theta(\bar{\omega})$ , vectors  $\mathbf{a}_3$  and  $\mathbf{a}^i$  ( $i = 1, 2, 3$ ), matrices  $\mathbf{A}_c$  and  $\mathbf{A}^c$  and the area element along  $S$  equal to  $\sqrt{a}d\mathbf{x}'$  (see Figure 5.1). In addition we assume that

$$\operatorname{ess\,inf}_{y \in \omega} \lambda_{\min}(\mathbf{A}^c(\mathbf{x}')), \quad \operatorname{ess\,inf}_{y \in \omega} \lambda_{\min}(\mathbf{A}_c(\mathbf{x}')) > 0. \quad (5.2.1)$$

In models of Naghdi's type (or in the case of the plane geometry Reissner-Mindlin's

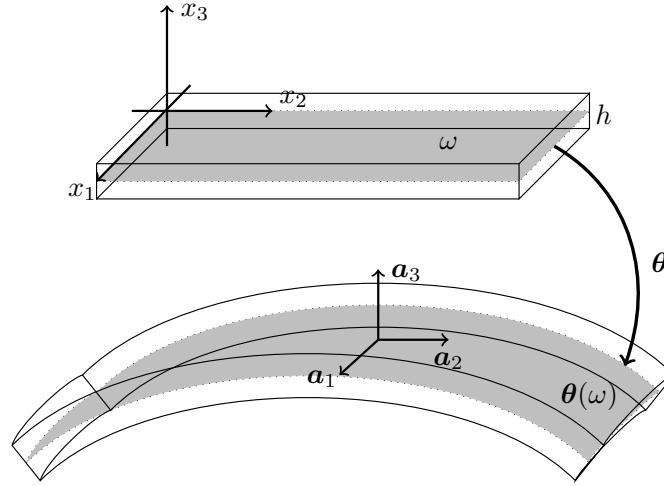


Figure 5.1: Parametrization of the undeformed shell.

type) in addition to the parametrization of the deformed shell  $\psi : \bar{\omega} \rightarrow \mathbb{R}^3$  there are independent functions that are used to describe shear of the cross-section of the shell with respect to the middle surface. In our case we will describe behavior of the cross-section using function  $\mathbf{S} : \bar{\omega} \rightarrow \text{SO}(3)$ . The relation of  $\psi$  and  $\mathbf{S}$  will not a priori be given as a restriction, but will be in a physical way penalized in the energy of the model.

For the surface parametrized by  $\psi$  we associate the notation

$$\mathbf{a}_3^\psi = \frac{\partial_1 \psi \times \partial_2 \psi}{\|\partial_1 \psi \times \partial_2 \psi\|}.$$

for the normal vector at the surface. Furthermore let  $\gamma_0 \subseteq \partial\omega$  be with positive capacity,

the set where the shell will be clamped. Then we define three sets of admissible functions

$$\begin{aligned} \mathcal{A}^N &= \{(\boldsymbol{\psi}, \mathbf{S}) \in W^{1,4}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3)) : \boldsymbol{\psi}|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0}, \mathbf{S}|_{\gamma_0} = \mathbf{I}, \\ &\quad \det \begin{bmatrix} \mathbf{S}\mathbf{a}_3 & \partial_1 \boldsymbol{\psi} & \partial_2 \boldsymbol{\psi} \end{bmatrix} \geq 0 \text{ a.e.}\}, \\ \mathcal{A}^K &= \{(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^N : \mathbf{S}\mathbf{a}_3 = \mathbf{a}_3^\psi, \mathbf{S}\mathbf{a}_1 \cdot \partial_2 \boldsymbol{\psi} = \mathbf{S}\mathbf{a}_2 \cdot \partial_1 \boldsymbol{\psi}\}, \\ \mathcal{A}^F &= \{(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^N : \mathbf{S}\mathbf{a}_\beta = \partial_\beta \boldsymbol{\psi}, \beta = 1, 2\}. \end{aligned} \quad (5.2.2)$$

The subscripts in these functions sets suggest that they correspond to the Naghdi, the Koiter and the flexural shell type models (see Section 5.3). On  $\mathcal{A}^N$  we will consider the functional of the following form

$$J(\boldsymbol{\psi}, \mathbf{S}) = h \int_{\omega} w_m(\mathbf{x}'; \mathbf{S}, \nabla' \boldsymbol{\psi}) \sqrt{a} d\mathbf{x}' + \frac{h^3}{12} \int_{\omega} q_f(\mathbf{x}'; \mathbf{S}, \nabla' \mathbf{S}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}'. \quad (5.2.3)$$

Here  $h$  stands for the thickness of the shell, the force surface density is for simplicity assumed to be independent of the deformation and given by  $\mathbf{f} \in L^{4/3}(\omega; \mathbb{R}^3)$  (since  $4/3$  is Hölder conjugate of 4), while  $w_m$  and  $q_f$  are membrane (and shear and drill) and flexural energy density functions incorporating all essential energies present in shell deformations. Notation  $\nabla' \mathbf{S}$  should usually be interpreted as block matrix function  $\begin{bmatrix} \partial_1 \mathbf{S} & \partial_2 \mathbf{S} \end{bmatrix}$ .

The nonlinear shell model of the Naghdi type we propose in this chapter is formulated as the following minimization problem:

$$\text{find } (\boldsymbol{\varphi}, \mathbf{R}) \in \mathcal{A}^N, \quad J(\boldsymbol{\varphi}, \mathbf{R}) = \inf_{(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^N} J(\boldsymbol{\psi}, \mathbf{S}). \quad (5.2.4)$$

The same problem can be also considered for subsets  $\mathcal{A}^K$  and  $\mathcal{A}^F$  which will lead us to a different type problems.

In order that the functional  $J$  captures membrane, shear, drill and flexural behavior of shell we have to specify particular energy density functions  $w_m$  and  $q_f$ . We assume

$$\begin{aligned} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}) &= W_m \left( \begin{bmatrix} (\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) \\ (\mathbf{S}\mathbf{a}_3)^T \nabla' \boldsymbol{\psi} \end{bmatrix} \right), \\ q_f(\mathbf{S}, \nabla' \mathbf{S}) &= Q_f(\mathbf{S}^T \nabla' \mathbf{S}) \end{aligned} \quad (5.2.5)$$

for some continuous functions  $Q_f : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$  and  $W_m : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$ . Note here that for each  $\beta \in \{1, 2\}$  the matrix  $\mathbf{S}^T \partial_\beta \mathbf{S}$  is skew symmetric and thus can be represented only

by three functions. Thus the matrix  $\mathbf{S}(\mathbf{x}')^T \nabla' \mathbf{S}(\mathbf{x}')$  we always consider as an element of  $\mathbb{R}^{3 \times 2}$ . Since in the energy there are two terms scaled differently with respect to the thickness  $h$  of the shell this kind of a shell model is not obtained as a limit model of the three-dimensional elasticity when the thickness tends to zero. However the choice of the energy densities is motivated by the obtained two-dimensional models in [51], for the membrane part, i.e.  $w_m$ , and [40] for the flexural part  $q_f$ . This will be clear in Section 5.4 when we consider asymptotic behavior of the functional (5.2.3) when  $h$  tends to zero. Also note that the strains do not change under the rigid deformations since  $\mathbf{S}^T \nabla' \boldsymbol{\psi}$  and  $\mathbf{S}^T \nabla' \mathbf{S}$  are independent of  $\mathbf{R} \in \text{SO}(3)$  and  $\mathbf{a} \in \mathbb{R}^3$  when we replace  $\mathbf{S}$  by  $\mathbf{R}\mathbf{S}$  and  $\boldsymbol{\psi}$  by  $\mathbf{R}\boldsymbol{\psi} + \mathbf{a}$ , i.e. the model is frame indifferent.

**Remark 5.2.1.** The strain in  $W_m$  can be also written by

$$(\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) = \mathbf{X}^T \mathbf{X} + 2 \nabla' \boldsymbol{\theta}^T \mathbf{X},$$

for  $\mathbf{X} = \mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}$ , so in general setting this matrix contains all information about membrane, shear and drill deformation.

Assumptions on functions  $Q_f, W_m$ :

$$\text{N1)} \quad \exists \mathcal{B}_f \in \mathcal{L}(\mathbb{R}^{3 \times 2}, \mathbb{R}^{3 \times 2}) \text{ such that } Q_f(\mathbf{X}) = \mathcal{B}_f \mathbf{X} \cdot \mathbf{X}, \forall \mathbf{X} \in \mathbb{R}^{3 \times 2},$$

$$\text{N2)} \quad \exists C > 0 \quad Q_f(\mathbf{X}) \leq C(1 + \|\mathbf{X}\|_F^2), \quad W_m(\mathbf{Y}) \leq C(1 + \|\mathbf{Y}\|_F^2), \quad \mathbf{X}, \mathbf{Y} \in \mathbb{R}^{3 \times 2},$$

$$\text{N3)} \quad \exists c > 0 \quad Q_f(\mathbf{X}) \geq c\|\mathbf{X}\|_F^2, \quad W_m(\mathbf{Y}) \geq c\|\mathbf{Y}\|_F^2, \quad \mathbf{X}, \mathbf{Y} \in \mathbb{R}^{3 \times 2}.$$

Assumptions on  $q_f$  are, for instance, clear consequences of  $\mathcal{B}_f$  being uniformly bounded and uniformly positive definite linear operator.

**Remark 5.2.2.** To be more specific we propose particular energy density functions that will lead us to the model for St. Venant–Kirchhoff material (STVK):

$$\begin{aligned} w_m^{STVK}(\mathbf{S}, \nabla' \boldsymbol{\psi}) &= \frac{1}{2} \mathcal{C}_m^2 \left( \begin{bmatrix} \frac{1}{2} \mathbf{A}^c (\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) \\ (\mathbf{S} \mathbf{a}_3)^T \nabla' \boldsymbol{\psi} \end{bmatrix} \right), \\ q_f^{STVK}(\mathbf{S}, \nabla' \mathbf{S}) &= \frac{1}{2} \mathcal{C}_f^2 \left( \frac{1}{\sqrt{a}} \begin{bmatrix} \mathbf{S}^T \partial_1 \mathbf{S} \mathbf{a}_2 \cdot \mathbf{a}_3 & \mathbf{S}^T \partial_2 \mathbf{S} \mathbf{a}_2 \cdot \mathbf{a}_3 \\ \mathbf{S}^T \partial_1 \mathbf{S} \mathbf{a}_3 \cdot \mathbf{a}_1 & \mathbf{S}^T \partial_2 \mathbf{S} \mathbf{a}_3 \cdot \mathbf{a}_1 \\ \mathbf{S}^T \partial_1 \mathbf{S} \mathbf{a}_1 \cdot \mathbf{a}_2 & \mathbf{S}^T \partial_2 \mathbf{S} \mathbf{a}_1 \cdot \mathbf{a}_2 \end{bmatrix} \right), \end{aligned} \quad (5.2.6)$$



where  $\mathcal{C}_m^2(\mathbf{X}) = \mathcal{C}_m \mathbf{X} \cdot \mathbf{X}$ ,  $\mathcal{C}_f^2(\mathbf{X}) = \mathcal{C}_f \mathbf{X} \cdot \mathbf{X}$  are quadratic in strains and the elasticity tensors  $\mathcal{C}_m, \mathcal{C}_f : \mathbb{M}^{3 \times 2} \rightarrow \mathbb{M}^{3 \times 2}$  are given by (1.3.2) In this case we obtain a model with linear constitutive equations with nonlinear strains.

### 5.3. PROPERTIES OF THE MODEL

#### 5.3.1. On sets of admissible functions

Besides of the smoothness assumptions and clamping boundary conditions at  $\gamma_0$ , the only constraint in  $\mathcal{A}^N$  is  $\det \begin{bmatrix} \mathbf{S}\mathbf{a}_3 & \partial_1\psi & \partial_2\psi \end{bmatrix} \geq 0$  which is related to the orientation preservation. Additional constraints in  $\mathcal{A}^K$  and  $\mathcal{A}^F$  are material constraints. Also note that the condition  $\det \begin{bmatrix} \mathbf{S}\mathbf{a}_3 & \partial_1\psi & \partial_2\psi \end{bmatrix} \geq 0$  is relevant only in  $\mathcal{A}^N$  since it is directly fulfilled in  $\mathcal{A}^F$  and  $\mathcal{A}^K$  since from condition  $\mathbf{a}_3^\psi = \mathbf{S}\mathbf{a}_3$  and the definition of  $\mathbf{a}_3^\psi$  we see that

$$\det \begin{bmatrix} \mathbf{S}\mathbf{a}_3 & \partial_1\psi & \partial_2\psi \end{bmatrix} = \det \begin{bmatrix} \mathbf{a}_3^\psi & \partial_1\psi & \partial_2\psi \end{bmatrix} > 0.$$

**Lemma 5.3.1.** It holds  $\mathcal{A}^F \subset \mathcal{A}^K \subset \mathcal{A}^N$ .

*Proof.* The last inclusion is obvious. To check the other inclusion, we can directly see that the condition  $\mathbf{S}\mathbf{a}_1 \cdot \partial_2\psi = \mathbf{S}\mathbf{a}_2 \cdot \partial_1\psi$  is satisfied for  $\begin{bmatrix} \partial_1\psi & \partial_2\psi \end{bmatrix} = \nabla'\psi = \mathbf{S}\nabla'\theta = \mathbf{S} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ . For the other condition:

$$\mathbf{a}_3^\psi = \frac{\partial_1\psi \times \partial_2\psi}{\|\partial_1\psi \times \partial_2\psi\|} = \frac{\mathbf{S}\mathbf{a}_1 \times \mathbf{S}\mathbf{a}_2}{\|\mathbf{S}\mathbf{a}_1 \times \mathbf{S}\mathbf{a}_2\|} = \mathbf{S} \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} = \mathbf{S}\mathbf{a}_3.$$

□

In the view of Lemma 5.5.3, it is possible to express conditions in  $\mathcal{A}^K$  in various ways:

$$\mathbf{S}\mathbf{a}_3 \cdot \partial_\beta\psi = 0, \beta = 1, 2 \quad \Longleftrightarrow \quad \mathbf{S}\mathbf{a}_\beta \cdot \mathbf{a}_3^\psi = 0, \beta = 1, 2 \quad \Longleftrightarrow \quad \mathbf{S}\mathbf{a}_3 = \mathbf{a}_3^\psi. \quad (5.3.1)$$

Therefore the condition  $\mathbf{S}\mathbf{a}_3 = \mathbf{a}_3^\psi$  from  $\mathcal{A}^K$  (and  $\mathcal{A}^F$ ) can be rephrased by that the cross-sections (described by  $\mathbf{a}_3$ ) after deformation  $\mathbf{S}\mathbf{a}_3$  remain normal to the deformed middle surface (whose tangent space is spanned by  $\partial_1\psi$  and  $\partial_2\psi$  (i.e. the shell is unshearable). Furthermore there is no extension in the normal direction since  $\mathbf{S}$  is unitary. This is typical for the Koiter-type models.

In  $\mathcal{A}^F$  we additionally have

$$\partial_{\beta_1}\psi \cdot \partial_{\beta_2}\psi = \mathbf{S}\partial_{\beta_1}\theta \cdot \mathbf{S}\partial_{\beta_2}\theta = \partial_{\beta_1}\theta \cdot \partial_{\beta_2}\theta$$

since  $\mathbf{S} \in \text{SO}(3)$ . This means that there is no change of metric tensor in  $\mathcal{A}^F$ , so  $\psi$  from  $\mathcal{A}^F$  are isometric deformations.

To analyse the energy functional (5.2.3) on the sets  $\mathcal{A}^F$  and  $\mathcal{A}^K$  we state the following obvious lemma.

**Lemma 5.3.2.** It holds

$$\begin{aligned} & \begin{bmatrix} \frac{1}{2}(\mathbf{S}^T \nabla' \psi + \nabla' \theta)^T (\mathbf{S}^T \nabla' \psi - \nabla' \theta) \\ (\mathbf{S} \mathbf{a}_3)^T \nabla' \psi \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(\nabla' \psi^T \nabla' \psi - \nabla' \theta^T \nabla' \theta) \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(\nabla' \psi^T \mathbf{S} \nabla' \theta - \nabla' \theta^T \mathbf{S}^T \nabla' \psi) \\ (\mathbf{S} \mathbf{a}_3)^T \nabla' \psi \end{bmatrix}. \quad (5.3.2) \end{aligned}$$

The matrices on the right hand side are orthogonal in the Frobenius scalar product.

*Proof.* Orthogonality follows since the first  $2 \times 2$  block is symmetric and the second one is skew-symmetric.  $\square$

This lemma implies that in the strain on the left hand side of (5.3.2) (with six components) the following is hidden:

- change of metric tensor (in the first term on the right hand side)
- the drill (in the upper  $2 \times 2$  skew-symmetric block of the second term on the right hand side)
- the shear (in the lower  $1 \times 2$  block in the second term on the right hand side).

### 5.3.2. The flexural shell model

We now observe the problem (5.2.4) set on the set of admissible functions  $\mathcal{A}^F$ . The model is actually well formulated for  $\theta \in W^{1,\infty}$  parametrizations, but for this comparison we assume more smoothness, namely  $\theta \in W^{2,\infty}(\omega; \mathbb{R}^3)$ .

In  $\mathcal{A}^F$  the strain in  $W_m$  is zero, which is directly obvious from the definition since  $\mathbf{S} \nabla' \theta = \nabla' \psi$  implies  $\mathbf{S}^T \nabla' \psi - \nabla' \theta = 0$  and Lemma 5.3.1 and (5.3.1). Thus only the term  $q_f$  from (5.2.3) is relevant for the problem.

In the matrix  $\mathbf{S}^T \nabla' \mathbf{S}$  some terms vanish as well. Firstly, we rewrite it in the following form

$$\mathbf{S}^T \partial_\beta \mathbf{S} = \sum_{i,j=1,2,3} (\mathbf{S}^T \partial_\beta \mathbf{S} \mathbf{a}_i \cdot \mathbf{a}_j) \mathbf{a}^i \otimes \mathbf{a}^j,$$

which holds since  $\mathbf{I}_3 = \sum_{i,j=1,2,3} \mathbf{a}_i \cdot \mathbf{a}_j \mathbf{a}^i \otimes \mathbf{a}^j$ .

Since  $\mathbf{S}$  is rotation,  $\mathbf{S}^T \partial_\beta \mathbf{S}$  is skew symmetric, so  $\mathbf{S}^T \partial_\beta \mathbf{S} \mathbf{a}_i \cdot \mathbf{a}_i = 0$  for  $i \in \{1, 2, 3\}$ . Moreover, since we have

$$\partial_2(\mathbf{a}_1) = \partial_{12} \boldsymbol{\theta} = \partial_1(\mathbf{a}_2) \quad \text{and} \quad \partial_2(\mathbf{S} \mathbf{a}_1) = \partial_{12} \boldsymbol{\psi} = \partial_1(\mathbf{S} \mathbf{a}_2), \quad (5.3.3)$$

we obtain

$$\begin{aligned} \mathbf{S}^T \partial_2 \mathbf{S} \mathbf{a}_1 \cdot \mathbf{a}_2 &= \partial_2 \mathbf{S} \mathbf{a}_1 \cdot \mathbf{S} \mathbf{a}_2 = (\partial_2(\mathbf{S} \mathbf{a}_1) - \mathbf{S} \partial_2 \mathbf{a}_1) \cdot \mathbf{S} \mathbf{a}_2 \\ &= (\partial_1(\mathbf{S} \mathbf{a}_2) - \mathbf{S} \partial_1 \mathbf{a}_2) \cdot \mathbf{S} \mathbf{a}_2 = \partial_1 \mathbf{S} \mathbf{a}_2 \cdot \mathbf{S} \mathbf{a}_2 = 0, \end{aligned}$$

where the last equality is from skew symmetricity of the matrix  $\mathbf{S}^T \partial_1 \mathbf{S}$ . We obtain similarly  $\mathbf{S}^T \partial_2 \mathbf{S} \mathbf{a}_2 \cdot \mathbf{a}_1 = 0$ , so the only terms left in the strains are  $\mathbf{S}^T \partial_{\beta_1} \mathbf{S} \mathbf{a}_3 \cdot \mathbf{a}_{\beta_2}$ . Moreover,

$$\begin{aligned} \mathbf{S}^T \partial_\beta \mathbf{S} \mathbf{a}_3 &= \mathbf{S}^T \partial_\beta \mathbf{S} \mathbf{a}_3 + \mathbf{S}^T \mathbf{S} \partial_\beta \mathbf{a}_3 - \partial_\beta \mathbf{a}_3 = \mathbf{S}^T \partial_\beta (\mathbf{S} \mathbf{a}_3) - \partial_\beta \mathbf{a}_3 = \mathbf{S}^T \partial_\beta (\mathbf{a}_3^\psi) - \partial_\beta \mathbf{a}_3 \\ &= S_M(\boldsymbol{\psi}) \mathbf{a}_\beta, \end{aligned}$$

where  $S_M$  is the shape mapping (Weingarten map). Furthermore, since  $\mathbf{S} \mathbf{a}_{\beta_2} = \mathbf{a}_{\beta_2}^\psi$

$$\begin{aligned} \mathbf{S}^T \partial_{\beta_1} \mathbf{S} \mathbf{a}_3 \cdot \mathbf{a}_{\beta_2} &= S_M(\boldsymbol{\psi}) \mathbf{a}_{\beta_1} \cdot \mathbf{a}_{\beta_2} = \mathbf{S}^T \partial_{\beta_1} \mathbf{a}_3^\psi \cdot \mathbf{a}_{\beta_2} - \partial_{\beta_1} \mathbf{a}_3 \cdot \mathbf{a}_{\beta_2} \\ &= \partial_{\beta_1} \mathbf{a}_3^\psi \cdot \mathbf{S} \mathbf{a}_{\beta_2} - \partial_{\beta_1} \mathbf{a}_3 \cdot \mathbf{a}_{\beta_2} = \partial_{\beta_1} \mathbf{a}_3^\psi \cdot \mathbf{a}_{\beta_2}^\psi - \partial_{\beta_1} \mathbf{a}_3 \cdot \mathbf{a}_{\beta_2} \\ &= b_{\beta_1 \beta_2}(\boldsymbol{\psi}) - b_{\beta_1 \beta_2}(\boldsymbol{\theta}), \end{aligned}$$

where  $\mathbf{b}(\boldsymbol{\psi}) = (b_{\beta_1 \beta_2}(\boldsymbol{\psi}))_{\beta_1, \beta_2}$  denote the curvature tensor of the surface parametrized by  $\boldsymbol{\psi}$ . Thus on  $\mathcal{A}^F$  the strain is the change of the curvature tensor. It typically appears in the flexural shell models.

All obtained expressions for  $\mathbf{S}^T \partial_\beta \mathbf{S} \mathbf{a}_i \cdot \mathbf{a}_j$  can be written in a matrix form. Let us for this purpose define

$$\mathbf{B}^\beta = \begin{bmatrix} 0 & 0 & -(b_{\beta 1}(\boldsymbol{\psi}) - b_{\beta 1}(\boldsymbol{\theta})) \\ 0 & 0 & -(b_{\beta 2}(\boldsymbol{\psi}) - b_{\beta 2}(\boldsymbol{\theta})) \\ b_{\beta 1}(\boldsymbol{\psi}) - b_{\beta 1}(\boldsymbol{\theta}) & b_{\beta 2}(\boldsymbol{\psi}) - b_{\beta 2}(\boldsymbol{\theta}) & 0 \end{bmatrix}, \quad (5.3.4)$$

for  $\beta = 1, 2$ . Then we have  $\mathbf{S}^T \partial_\beta \mathbf{S} \mathbf{a}_i \cdot \mathbf{a}_j = B_{i,j}^\beta$ , for  $i, j = 1, 2, 3$ . Also, using notations  $\mathbf{Q} = \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix}$  and  $\mathbf{B} := \begin{bmatrix} \mathbf{B}^1 & \mathbf{B}^2 \end{bmatrix}$ , we can also write

$$\mathbf{S}^T \partial_\beta \mathbf{S} = \sum_{i,j=1,2,3} (\mathbf{S}^T \partial_\beta \mathbf{S} \mathbf{a}_i \cdot \mathbf{a}_j) \mathbf{a}^i \otimes \mathbf{a}^j = \sum_{i,j=1,2,3} B_{i,j}^\beta (\mathbf{Q} e_j) (\mathbf{Q} e_i)^T = \mathbf{Q} \mathbf{B}^\beta \mathbf{Q}^T, \quad (5.3.5)$$

and consequently

$$\mathbf{S}^T \nabla' \mathbf{S} = \mathbf{Q} \mathbf{B} \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{Q} \end{bmatrix}^T. \quad (5.3.6)$$

Let us define an operator  $\mathcal{P} : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{M}^{3 \times 6}$  by

$$\mathcal{P} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & -a & 0 & 0 & -c \\ 0 & 0 & -b & 0 & 0 & -d \\ a & b & 0 & c & d & 0 \end{bmatrix}. \quad (5.3.7)$$

By its definition, we have  $\mathcal{P}(\mathbf{b}(\boldsymbol{\psi}) - \mathbf{b}(\boldsymbol{\theta})) = \mathbf{B}$ . Finally, we can define a function  $\tilde{q}_f : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$  (obtained from  $Q_f$  by change of coordinates and incorporating terms from undeformed geometry) by

$$\tilde{q}_f(\mathbf{X}) = Q_f \left( \mathbf{Q} \mathcal{P}(\mathbf{X}) \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{Q} \end{bmatrix}^T \right). \quad (5.3.8)$$

From above it is clear that satisfies

$$\tilde{q}_f(\mathbf{b}(\boldsymbol{\psi}) - \mathbf{b}(\boldsymbol{\theta})) = Q_f(\mathbf{S}^T \nabla' \mathbf{S}) = q_f(\mathbf{S}, \nabla' \mathbf{S}). \quad (5.3.9)$$

Therefore, the problem (5.2.4) on  $\mathcal{A}^F$  can be stated as minimization of the functional

$$J(\boldsymbol{\psi}, \mathbf{S}) = \frac{h^3}{12} \int_{\omega} \tilde{q}_f(\mathbf{b}(\boldsymbol{\psi}) - \mathbf{b}(\boldsymbol{\theta})) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}', \quad (5.3.10)$$

which is a functional of the same form as obtained in [40]. Thus since the set  $\mathcal{A}^F$  is a set of inextensional deformations minimization of the functional  $J$  from (5.2.3) on  $\mathcal{A}^F$  is exactly the same as the flexural shell model from [40]. Note however that for the formulation (5.3.10) at least  $W^{2,\infty}$  geometry is necessary and only  $W^{1,\infty}$  for the formulation (5.2.3) on  $\mathcal{A}^F$ .

**Remark 5.3.3.** One can see that if we had  $\nabla' \boldsymbol{\psi} = -\mathbf{S} \nabla' \boldsymbol{\theta}$  as the condition in  $\mathcal{A}^F$  instead of  $\nabla' \boldsymbol{\psi} = \mathbf{S} \nabla' \boldsymbol{\theta}$ , all the calculations stay the same.

**Remark 5.3.4.** Moreover, in the case of St. Venant–Kirchhoff material, the appropriate function  $\tilde{q}_f$  from [40] is exactly the tensor  $\mathcal{A}$  from (1.3.3). By direct calculations as above, this is equal to  $q_f^{STVK}(\mathbf{S}, \nabla' \mathbf{S})$ , where  $q_f^{STVK}$  is from (5.2.6). So, in the St. Venant–Kirchhoff material functionals our functional (5.2.3) on  $\mathcal{A}^F$  and the functional from [40] are not only of the same form, but they coincide.

**Remark 5.3.5.** In the case of the planar geometry for the middle surface of undeformed shell ( $\boldsymbol{\theta} = (\mathbf{x}', 0)$ ), since  $\mathbf{b}(\boldsymbol{\theta}) = 0$ , we can also write

$$\tilde{q}_f(\mathbf{b}(\boldsymbol{\psi}) - \mathbf{b}(\boldsymbol{\theta})) = \tilde{q}_f(\mathbf{b}(\boldsymbol{\psi})).$$

### 5.3.3. The Koiter shell model

According to (5.3.2), the membrane strain reduces to

$$\begin{bmatrix} \frac{1}{2}(\nabla' \boldsymbol{\psi}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}) \\ 0 \end{bmatrix}. \quad (5.3.11)$$

This is usual membrane strain, see e.g. [21]. Note that

$$\nabla' \boldsymbol{\psi}^T \nabla' \boldsymbol{\psi} = (\mathbf{a}_{\beta_1}^{\boldsymbol{\psi}} \cdot \mathbf{a}_{\beta_2}^{\boldsymbol{\psi}})_{\beta_1, \beta_2} = a_{\beta_1 \beta_2}(\boldsymbol{\psi}) = \mathbf{a}(\boldsymbol{\psi})$$

is the metric tensor of the surface parametrized by  $\boldsymbol{\psi}$ . Thus the above measure of membrane deformation is given as the change of the metric tensor, i.e. the difference of metric tensors on deformed and undeformed deformations  $\mathbf{a}(\boldsymbol{\psi}) - \mathbf{a}(\boldsymbol{\theta})$ .

Moreover, since  $(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^K$  we are in the assumptions of Lemma 5.5.4 for  $P_\beta = \partial_\beta \boldsymbol{\psi}$ ,  $\beta = 1, 2$  and  $M_\beta = \partial_\beta \boldsymbol{\theta}$ . Thus according to Remark 5.5.5, there are only two pointwise choices for the matrix  $\mathbf{S}$ . However assumed regularity of  $\mathbf{S}$  implies that there are only two choices for  $\mathbf{S}$  but as a function. Now the boundary condition on  $\mathbf{S}$  implies that  $\mathbf{S}$  is uniquely defined by terms of  $\nabla' \boldsymbol{\psi}$  and  $\nabla' \boldsymbol{\theta}$ . We can check that we still have (since for this we only use  $\mathbf{S} \mathbf{a}_3 = \mathbf{a}_3^{\boldsymbol{\psi}}$ )

$$\mathbf{S}^T \partial_\beta \mathbf{S} \mathbf{a}_3 = S_{M, \mathbf{u}}(\boldsymbol{\psi}) \mathbf{a}_\beta. \quad (5.3.12)$$

However this now leads to

$$\begin{aligned}
 \mathbf{S}^T \partial_{\beta_1} \mathbf{S} \mathbf{a}_3 \cdot \mathbf{a}_{\beta_2} &= S_{M,u}(\boldsymbol{\psi}) \mathbf{a}_{\beta_1} \cdot \mathbf{a}_{\beta_2} = \mathbf{S}^T \partial_{\beta_1} \mathbf{a}_3^\psi \cdot \mathbf{a}_{\beta_2} - \partial_{\beta_1} \mathbf{a}_3 \cdot \mathbf{a}_{\beta_2} \\
 &= \partial_{\beta_1} \mathbf{a}_3^\psi \cdot \mathbf{S} \mathbf{a}_{\beta_2} - \partial_{\beta_1} \mathbf{a}_3 \cdot \mathbf{a}_{\beta_2} \\
 &= \partial_{\beta_1} \mathbf{a}_3^\psi \cdot \mathbf{a}_{\beta_2}^\psi - \partial_{\beta_1} \mathbf{a}_3 \cdot \mathbf{a}_{\beta_2} + \partial_{\beta_1} \mathbf{a}_3^\psi \cdot (\mathbf{S} \mathbf{a}_{\beta_2} - \mathbf{a}_{\beta_2}^\psi) \\
 &= b_{\beta_1 \beta_2}(\boldsymbol{\psi}) - b_{\beta_1 \beta_2} + \partial_{\beta_1} \mathbf{a}_3^\psi \cdot (\mathbf{S} \mathbf{a}_{\beta_2} - \mathbf{a}_{\beta_2}^\psi).
 \end{aligned}$$

Now since  $\mathbf{a}_3^\psi \cdot \mathbf{S} \mathbf{a}_{\beta_2} = 0$  and  $\mathbf{a}_3^\psi \cdot \mathbf{a}_{\beta_2}^\psi = 0$  we obtain that

$$\partial_{\beta_1} \mathbf{a}_3^\psi \cdot (\mathbf{S} \mathbf{a}_{\beta_2} - \mathbf{a}_{\beta_2}^\psi) = -\mathbf{a}_3^\psi \cdot \partial_{\beta_1} (\mathbf{S} \mathbf{a}_{\beta_2} - \mathbf{a}_{\beta_2}^\psi) = -\mathbf{S} \mathbf{a}_3 \cdot \partial_{\beta_1} (\mathbf{S} \mathbf{a}_{\beta_2} - \partial_{\beta_2} \boldsymbol{\psi}). \quad (5.3.13)$$

Furthermore

$$\begin{aligned}
 \mathbf{S}^T \partial_2 \mathbf{S} \mathbf{a}_1 \cdot \mathbf{a}_2 &= \partial_2 \mathbf{S} \mathbf{a}_1 \cdot \mathbf{S} \mathbf{a}_2 = (\partial_2 (\mathbf{S} \mathbf{a}_1) - \mathbf{S} \partial_2 \mathbf{a}_1) \cdot \mathbf{S} \mathbf{a}_2 = (\partial_2 (\mathbf{S} \mathbf{a}_1) - \mathbf{S} \partial_1 \mathbf{a}_2) \cdot \mathbf{S} \mathbf{a}_2 \\
 &= (\partial_2 (\mathbf{S} \mathbf{a}_1) - \partial_1 (\mathbf{S} \mathbf{a}_2) + \partial_1 \mathbf{S} \mathbf{a}_2) \cdot \mathbf{S} \mathbf{a}_2 = (\partial_2 (\mathbf{S} \mathbf{a}_1) - \partial_1 (\mathbf{S} \mathbf{a}_2)) \cdot \mathbf{S} \mathbf{a}_2 \\
 &= (\partial_2 (\mathbf{S} \mathbf{a}_1 - \partial_1 \boldsymbol{\psi}) - \partial_1 (\mathbf{S} \mathbf{a}_2 - \partial_2 \boldsymbol{\psi})) \cdot \mathbf{S} \mathbf{a}_2.
 \end{aligned} \quad (5.3.14)$$

Now we see from the from (5.3.13) and (5.3.14) that the flexural part of the strain is given as a perturbation of the change of the curvature tensor of the order  $h^3$  of the derivatives of the membrane strains which in the energy contribute by the factor  $h$ . Now the energy of the nonlinear model of the Koiter type following from  $J$  on  $\mathcal{A}^K$  is given by

$$\begin{aligned}
 J(\boldsymbol{\psi}, \mathbf{S}) &= \frac{1}{2} h \int_{\omega} W_m(\mathbf{a}(\boldsymbol{\psi}) - \mathbf{a}(\boldsymbol{\theta})) \sqrt{a} d\mathbf{x}' \\
 &\quad + \frac{h^3}{12} \int_{\omega} \tilde{q}_f(\mathbf{b}(\boldsymbol{\psi}) - \mathbf{b}(\boldsymbol{\theta}) + \text{perturbation}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}'.
 \end{aligned} \quad (5.3.15)$$

For a particular choice of the membrane energy density function  $W_m$  as in the case of the St. Venant–Kirchhoff material in (5.2.6) the membrane energy is equal to the classical Koiter model (see [21, 47]). As already noted the flexural strain is a perturbation of the classical strain given as the change of curvature tensor. Similar situation occurred in the linear models in [86] and [87]. In [47] Koiter argues that the theory obtained using lower order perturbations are equivalent: ”We have stressed repeatedly that the addition of terms of this type to the strain energy per unit area is often convenient in order to obtain the equations in the simplest possible form for the problem at hand.”

## 5.3.4. Linearization of the model

In this subsection we linearize the model given in (5.2.4). We first linearize the nonlinear strains from (5.2.5) and compare in Proposition 5.3.6 the obtained energy for the St. Venant–Kirchhoff material as in (5.2.6). In the second step in Proposition 5.3.7 we linearize the conditions in the sets of admissible functions. The obtained linearized strain, the total energy functional and the constraints in the function spaces turn out to be exactly the same as in the linear Naghdi type model from [87] (and presented in Chapter 1).

**Proposition 5.3.6.** Linearization of nonlinear strains given in (5.2.5) gives strains as in the linear model of Naghdi type from [87]. Furthermore linearization of the nonlinear STVK energy as given in Remark 5.2.2 gives energy as in the linear model of Naghdi type from [87].

*Proof.* For both, it is sufficient to prove that

$$\begin{aligned} & \begin{bmatrix} \frac{1}{2} \mathbf{A}^c (\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) \\ (\mathbf{S} \mathbf{a}_3)^T \nabla' \boldsymbol{\psi} \end{bmatrix} \\ & \approx \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix}^T \begin{bmatrix} \partial_1 \mathbf{u} + \mathbf{a}_1 \times \boldsymbol{\omega} & \partial_2 \mathbf{u} + \mathbf{a}_2 \times \boldsymbol{\omega} \end{bmatrix} \end{aligned} \quad (5.3.16)$$

and

$$\frac{1}{\sqrt{a}} \begin{bmatrix} \mathbf{S}^T \partial_1 \mathbf{S} \mathbf{a}_2 \cdot \mathbf{a}_3 & \mathbf{S}^T \partial_2 \mathbf{S} \mathbf{a}_2 \cdot \mathbf{a}_3 \\ \mathbf{S}^T \partial_1 \mathbf{S} \mathbf{a}_3 \cdot \mathbf{a}_1 & \mathbf{S}^T \partial_2 \mathbf{S} \mathbf{a}_3 \cdot \mathbf{a}_1 \\ \mathbf{S}^T \partial_1 \mathbf{S} \mathbf{a}_1 \cdot \mathbf{a}_2 & \mathbf{S}^T \partial_2 \mathbf{S} \mathbf{a}_1 \cdot \mathbf{a}_2 \end{bmatrix} \approx \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix}^T \nabla' \boldsymbol{\omega}. \quad (5.3.17)$$

Since  $\mathbf{S}(\mathbf{x}') \in \text{SO}(3)$  a.e., there is a skew symmetric matrix  $\mathbf{A}_\omega$  ( $\boldsymbol{\omega}$  denotes its axial vector) such that

$$\mathbf{S} = \exp(\mathbf{A}_\omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}_\omega^k \approx \mathbf{I} + \mathbf{A}_\omega.$$

In linearized elasticity we usually use displacement  $\mathbf{u}$  as the difference of the deformed and undeformed geometry:  $\mathbf{u} = \boldsymbol{\psi} - \boldsymbol{\theta}$ . Now we have

$$\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta} \approx (\mathbf{I} - \mathbf{A}_\omega) (\nabla' \boldsymbol{\theta} + \nabla' \mathbf{u}) - \nabla' \boldsymbol{\theta} \approx \nabla' \mathbf{u} - \mathbf{A}_\omega \nabla' \boldsymbol{\theta}. \quad (5.3.18)$$

Similarly,  $\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta} \approx \nabla' \mathbf{u} - \mathbf{A}_\omega \nabla' \boldsymbol{\theta} + 2 \nabla' \boldsymbol{\theta}$ . Linearization of their product is

$$\begin{aligned} & (\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) \approx 2 \nabla' \boldsymbol{\theta}^T (\nabla' \mathbf{u} - \mathbf{A}_\omega \nabla' \boldsymbol{\theta}) \\ & = 2 \nabla' \boldsymbol{\theta}^T \begin{bmatrix} \partial_1 \mathbf{u} + \mathbf{a}_1 \times \boldsymbol{\omega} & \partial_2 \mathbf{u} + \mathbf{a}_2 \times \boldsymbol{\omega} \end{bmatrix}. \end{aligned} \quad (5.3.19)$$



Moreover (since  $\mathbf{a}_3^T \nabla' \boldsymbol{\theta} = 0$ ):

$$\begin{aligned}
 (\mathbf{S} \mathbf{a}_3)^T \nabla' \boldsymbol{\psi} &\approx ((\mathbf{I} + \mathbf{A}_\omega) \mathbf{a}_3)^T (\nabla' \boldsymbol{\theta} + \nabla' \mathbf{u}) \\
 &= \mathbf{a}_3^T \nabla' \boldsymbol{\theta} + \mathbf{a}_3^T \nabla' \mathbf{u} + (\boldsymbol{\omega} \times \mathbf{a}_3)^T \nabla' \boldsymbol{\theta} + (\boldsymbol{\omega} \times \mathbf{a}_3)^T \nabla' \mathbf{u} \\
 &\approx \mathbf{a}_3^T \nabla' \mathbf{u} + (\boldsymbol{\omega} \times \mathbf{a}_3)^T \nabla' \boldsymbol{\theta} = \mathbf{a}_3^T \begin{bmatrix} \partial_1 \mathbf{u} + \mathbf{a}_1 \times \boldsymbol{\omega} & \partial_2 \mathbf{u} + \mathbf{a}_2 \times \boldsymbol{\omega} \end{bmatrix}.
 \end{aligned} \tag{5.3.20}$$

Since

$$\begin{aligned}
 \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix}^T &= \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix}^T \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}^T \\
 &= \begin{bmatrix} \mathbf{A}^c & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}^T, \tag{5.3.21}
 \end{aligned}$$

we have the first claim. For the second

$$\mathbf{S}^T \partial_\beta \mathbf{S} \mathbf{a}_i \cdot \mathbf{a}_j = \partial_\beta \mathbf{S} \mathbf{a}_i \cdot \mathbf{S}^T \mathbf{a}_j \approx \mathbf{A}_{\partial_\beta \boldsymbol{\omega}} \mathbf{a}_i \cdot (\mathbf{I} + \mathbf{A}_\omega) \mathbf{a}_j \approx \mathbf{A}_{\partial_\beta \boldsymbol{\omega}} \mathbf{a}_i \cdot \mathbf{a}_j = (\mathbf{a}_i \times \mathbf{a}_j) \cdot \partial_\beta \boldsymbol{\omega}. \tag{5.3.22}$$

Since from (1.2.1) we have  $\mathbf{a}^1 = \frac{1}{\sqrt{a}}(\mathbf{a}_2 \times \mathbf{a}_3)$ ,  $\mathbf{a}^2 = \frac{1}{\sqrt{a}}(\mathbf{a}_3 \times \mathbf{a}_1)$ ,  $\mathbf{a}^3 = \frac{1}{\sqrt{a}}(\mathbf{a}_1 \times \mathbf{a}_2)$ , by checking each coordinate, we see that the second claim is also fulfilled.  $\square$

**Proposition 5.3.7.** Linearization of the conditions in sets  $\mathcal{A}^N$ ,  $\mathcal{A}^K$ ,  $\mathcal{A}^F$  gives conditions in spaces  $V_N$ ,  $V_K$ ,  $V_F$ , respectively, the spaces corresponding to the spaces in the linear Naghdi type models from [87].

*Proof.* For  $\mathcal{A}^F$  we see that

$$(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^F \iff \mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta} = \mathbf{0}, \quad (\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^N. \tag{5.3.23}$$

and we proved

$$\mathbf{S}^T \nabla' \boldsymbol{\psi} - \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \approx \begin{bmatrix} \partial_1 \mathbf{u} + \mathbf{a}_1 \times \boldsymbol{\omega} & \partial_2 \mathbf{u} + \mathbf{a}_2 \times \boldsymbol{\omega} \end{bmatrix}.$$

For  $\mathcal{A}^K$  we use (5.3.2) to see that  $(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^K$  if and only if

$$\begin{bmatrix} \frac{1}{2}(\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) \\ (\mathbf{S} \mathbf{a}_3)^T \nabla' \boldsymbol{\psi} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\nabla' \boldsymbol{\psi}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}) \\ 0 \end{bmatrix}. \tag{5.3.24}$$

The linearization of the matrix on the right hand side is given by

$$\begin{aligned}
 \frac{1}{2} (\nabla' \boldsymbol{\psi}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}) &= \frac{1}{2} ((\nabla' \boldsymbol{\theta} + \nabla' \mathbf{u})^T (\nabla' \boldsymbol{\theta} + \nabla' \mathbf{u}) - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}) \\
 &\approx \frac{1}{2} (\nabla' \boldsymbol{\theta}^T \nabla' \mathbf{u} + \nabla' \mathbf{u}^T \nabla' \boldsymbol{\theta}) = \gamma(\mathbf{u})
 \end{aligned}$$

and is known as linearization of the change of metric tensor and usually appears in the linear membrane, flexural and Koiter shell models. Then from the proof of Proposition 5.3.6 we obtain linearization of the matrix on the left hand side and obtain

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}^T \begin{bmatrix} \partial_1 \mathbf{u} + \boldsymbol{\omega} \times \mathbf{a}_1 & \partial_2 \mathbf{u} + \boldsymbol{\omega} \times \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \gamma(\mathbf{u}) \\ 0 \end{bmatrix}, \quad (5.3.25)$$

which is equivalent to  $(\mathbf{u}, \boldsymbol{\omega}) \in V_K$  due to Lemma 4.1. in [87].  $\square$

Note that condition  $\det \begin{bmatrix} \mathbf{S}\mathbf{a}_3 & \partial_1 \boldsymbol{\psi} & \partial_2 \boldsymbol{\psi} \end{bmatrix} \geq 0$  from  $\mathcal{A}^N$  after linearization implies the condition  $1 + \sqrt{a}(\mathbf{a}^1 \cdot \partial_1 \mathbf{u} + \mathbf{a}^2 \cdot \partial_2 \mathbf{u}) \geq 0$ , which is inherently fulfilled for small deformations in linear theories and leaves no condition in the function space.

### 5.3.5. Differential formulation

In this subsection we assume that the solution of (5.2.4) that is regular enough exists and derive the differential equations.

In order to obtain the weak and differential formulation of the model we need to find the Gâteaux derivative of the functional  $J$  over  $\mathcal{A}^N$ . Let  $(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^N$  minimizes the total energy functional (5.2.4) and satisfies  $\det \begin{bmatrix} \mathbf{S}\mathbf{a}_3 & \partial_1 \boldsymbol{\psi} & \partial_2 \boldsymbol{\psi} \end{bmatrix} > 0$ . The last condition on the determinant is additional assumption in order to derive the differential equations of the model. Let  $\varepsilon > 0$  and let the perturbation  $(\boldsymbol{\psi}_\varepsilon, \mathbf{S}_\varepsilon) \in \mathcal{A}^N$  of  $(\boldsymbol{\psi}, \mathbf{S})$  is in the following form

$$\boldsymbol{\psi}_\varepsilon = \boldsymbol{\psi} + \varepsilon \mathbf{v}, \quad \mathbf{S}_\varepsilon = e^{\varepsilon \mathbf{A}_w} \mathbf{S},$$

for some smooth enough functions  $\mathbf{v}, \mathbf{w} : \omega \rightarrow \mathbb{R}^3$  that satisfy boundary conditions  $\mathbf{v}|_{\gamma_0} = \mathbf{w}|_{\gamma_0} = 0$ . Here  $\mathbf{A}_w$  is the skew-symmetric matrix of order 3 with axial vector  $\mathbf{w}$ . Then we obtain

$$\begin{aligned} (\mathbf{S}_\varepsilon^T \nabla' \boldsymbol{\psi}_\varepsilon + \nabla' \boldsymbol{\theta})^T (\mathbf{S}_\varepsilon^T \nabla' \boldsymbol{\psi}_\varepsilon - \nabla' \boldsymbol{\theta}) &= (\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) \\ &+ \varepsilon \left( (\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T \mathbf{S}^T (\nabla' \mathbf{v} - \mathbf{A}_w \nabla' \boldsymbol{\psi}) \right. \\ &\quad \left. + (\nabla' \mathbf{v} - \mathbf{A}_w \nabla' \boldsymbol{\psi})^T \mathbf{S} (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) \right) + O(\varepsilon^2), \end{aligned}$$

$$(\mathbf{S}_\varepsilon \mathbf{a}_3)^T \nabla' \boldsymbol{\psi}_\varepsilon = (\mathbf{S} \mathbf{a}_3)^T \nabla' \boldsymbol{\psi} + \varepsilon (\mathbf{S} \mathbf{a}_3)^T (\nabla' \mathbf{v} - \mathbf{A}_w \nabla' \boldsymbol{\psi}) + O(\varepsilon^2),$$

$$\mathbf{S}_\varepsilon^T \partial_\beta \mathbf{S}_\varepsilon = \mathbf{S}^T \partial_\beta \mathbf{S} + \varepsilon \mathbf{S}^T \mathbf{A}_{\partial_\beta \mathbf{w}} \mathbf{S} + O(\varepsilon^2) = \mathbf{S}^T \partial_\beta \mathbf{S} + \varepsilon \mathbf{A}_{\mathbf{S}^T \partial_\beta \mathbf{w}} + O(\varepsilon^2), \quad \beta = 1, 2.$$

Next we plug this perturbation into functional  $J$  from (5.2.3) with the energy density functions given by (5.2.5). Then the stationary point of the functional satisfies the following equation

$$\begin{aligned} h \int_{\omega} W'_m \left( \begin{bmatrix} (\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) \\ (\mathbf{S} \mathbf{a}_3)^T \nabla' \boldsymbol{\psi} \end{bmatrix} \right) \\ \cdot \begin{bmatrix} (\nabla' \boldsymbol{\psi} + \mathbf{S} \nabla' \boldsymbol{\theta})^T (\nabla' \mathbf{v} - \mathbf{A}_w \nabla' \boldsymbol{\psi}) + (\nabla' \mathbf{v} - \mathbf{A}_w \nabla' \boldsymbol{\psi})^T (\nabla' \boldsymbol{\psi} - \mathbf{S} \nabla' \boldsymbol{\theta}) \\ (\mathbf{S} \mathbf{a}_3)^T (\nabla' \mathbf{v} - \mathbf{A}_w \nabla' \boldsymbol{\psi}) \end{bmatrix} \sqrt{a} d\mathbf{x}' \\ + \frac{h^3}{12} \int_{\omega} Q'_f (\mathbf{S}^T \nabla' \mathbf{S}) \cdot \begin{bmatrix} \mathbf{A}_{\mathbf{S}^T \partial_1 w} & \mathbf{A}_{\mathbf{S}^T \partial_2 w} \end{bmatrix} \sqrt{a} d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \mathbf{v} \sqrt{a} d\mathbf{x}', \end{aligned} \quad (5.3.26)$$

for all smooth functions  $\mathbf{v}$  and  $\mathbf{w}$ . Since  $\mathbf{A}_{\mathbf{S}^T \partial_{\beta} w}$ ,  $\beta = 1, 2$  are skew-symmetric matrices only skew-symmetric parts of sub-matrices of  $3 \times 6$  matrix  $Q'_f (\mathbf{S}^T \nabla' \mathbf{S})$  are relevant in the flexural energy. Thus there are matrix functions  $\hat{\mathbf{n}}, \hat{\mathbf{m}}$  of type  $3 \times 2$  such that (5.3.26) can be written by

$$\int_{\omega} \hat{\mathbf{n}} \cdot \left( \nabla' \mathbf{v} + \begin{bmatrix} \partial_1 \boldsymbol{\psi} \times \mathbf{w} & \partial_2 \boldsymbol{\psi} \times \mathbf{w} \end{bmatrix} \right) \sqrt{a} d\mathbf{x}' + \int_{\omega} \hat{\mathbf{m}} \cdot \nabla' \mathbf{w} \sqrt{a} d\mathbf{x}' = \int_{\omega} \mathbf{f} \cdot \mathbf{v} \sqrt{a} d\mathbf{x}'. \quad (5.3.27)$$

From this equation we can derive the differential equations of the model.

$$\begin{aligned} \frac{1}{\sqrt{a}} \operatorname{div} (\sqrt{a} \hat{\mathbf{n}}) + \mathbf{f} &= 0, \\ \frac{1}{\sqrt{a}} \operatorname{div} (\sqrt{a} \hat{\mathbf{m}}) + \partial_1 \boldsymbol{\psi} \times \hat{\mathbf{n}}_1 + \partial_2 \boldsymbol{\psi} \times \hat{\mathbf{n}}_2 &= 0, \end{aligned} \quad (5.3.28)$$

where  $\hat{\mathbf{n}}_1$  and  $\hat{\mathbf{n}}_2$  are columns of the stress tensor  $\hat{\mathbf{n}} = \begin{bmatrix} \hat{\mathbf{n}}_1 & \hat{\mathbf{n}}_2 \end{bmatrix}$ . These equations are exactly the equations of the special Cosserat shell model with a single director, see [6]. The first equation represents the equation of equilibrium of forces and the second equilibrium of moments ( $\hat{\mathbf{n}}$  and  $\hat{\mathbf{m}}$  are the force and couple stress tensors). The dependance of  $\hat{\mathbf{n}}$  and  $\hat{\mathbf{m}}$  on the variables  $\mathbf{S}$ ,  $\nabla' \boldsymbol{\psi}$  and  $\nabla' \mathbf{S}$  represents particular constitutive law. The linear Naghdi type model from [87] has a similar structure and the equilibrium equations are the same as in (5.3.28) with  $\partial_{\beta} \boldsymbol{\psi}$  replaced by  $\partial_{\beta} \boldsymbol{\theta}$ , for  $\beta \in \{1, 2\}$ .

## 5.4. CONVERGENCE IN REGIMES

In this section we analyse behavior of the total energy functional (5.2.3) using the  $\Gamma$ -convergence with respect to the small parameter  $h$  (thickness). We do it in various regimes with the same scaling the Lamé coefficients (implicitly given in functions  $w_m$  and  $q_f$ ) as it was done when the corresponding regimes were analysed starting from the three-dimensional nonlinear elasticity in Chapter 3. More precisely, we observe the  $\Gamma$ -limit of the family of functionals  $(J_h^{2d,\alpha}(\boldsymbol{\psi}, \mathbf{S}))_{h>0}$ , where  $J_h^{2d,\alpha} : L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3)) \rightarrow \overline{\mathbb{R}}$  are defined by

$$J_h^{2d,\alpha}(\boldsymbol{\psi}, \mathbf{S}) = \begin{cases} h^{-\alpha} \int_{\omega} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}) \sqrt{a} d\mathbf{x}' \\ \quad + \frac{h^{\omega-2-\alpha}}{12} \int_{\omega} q_f(\mathbf{S}, \nabla' \mathbf{S}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}' & (\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^N, \\ +\infty & \text{otherwise.} \end{cases} \quad (5.4.1)$$

Thus, in functional  $J$  from (5.2.3) we incorporated the effect of elasticity coefficients in functions  $w_m$  and  $q_f$  (without the change of their notation) by explicitly pulling the factor  $h^{-\alpha-1}$  out of those functions, analogously to (3.1.1). The force term is kept unscaled.

Our goal is to prove a convergence result in various regimes for  $\alpha > -1$ , similar to Theorem 3.2.1 where we formulated known results of the asymptotic analysis of thin three-dimensional bodies in the case of planar middle surface (geometry). We will seek for the  $\Gamma$ -limits of the family of functionals  $(J_h^{2d,\alpha})_{h>0}$  in the strong  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology, but due to reasons similar to the ones stated in Remark 3.1.2, in view of Lemma 5.4.3, it is equivalent to seeking  $\Gamma$ -limits in the weak  $W^{1,4}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3))$  topology.

Due to technical reasons, we will also observe family of functionals  $(J_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  for which we have  $J_{\mathbf{S},h}^{2d,\alpha}(\boldsymbol{\psi}) = \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} J_h^{2d,\alpha}(\boldsymbol{\psi}, \mathbf{S})$  for all  $h > 0$  and  $\alpha > -1$ . More

precisely: we define  $J_{\mathbf{S},h}^{2d,\alpha} : L^4(\omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  by

$$J_{\mathbf{S},h}^{2d,\alpha}(\psi) = \begin{cases} \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} \left[ h^{-\alpha} \int_{\omega} w_m(\mathbf{S}, \nabla' \psi) \sqrt{a} d\mathbf{x}' \right. \\ \quad \left. + \frac{h^{2-\alpha}}{12} \int_{\omega} q_f(\mathbf{S}, \nabla' \mathbf{S}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}' \right] & \psi \in \mathcal{A}_{\mathbf{S}}^N, \\ +\infty & \text{otherwise,} \end{cases} \quad (5.4.2)$$

where

$$\mathcal{A}_{\mathbf{S}}^N := \{ \psi \in W^{1,4}(\omega; \mathbb{R}^3) : \exists \mathbf{S} \in W^{1,2}(\omega; \text{SO}(3)) \text{ s.t. } (\psi, \mathbf{S}) \in \mathcal{A}^N \}. \quad (5.4.3)$$

Note that even though the infimum for  $\mathbf{S}$  in definition of the functional  $J_{\mathbf{S},h}^{2d,\alpha}$  is taken in the set  $L^2(\omega; \text{SO}(3))$ , effectively due to the definition of the set of admissible functions for the functional  $J_h^{2d,\alpha}$  this infimum is taken over the set  $W^{1,2}(\omega; \text{SO}(3))$ . We will seek the  $\Gamma$ -limit of this family of functionals  $(J_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  in the strong  $L^4(\omega; \mathbb{R}^3)$  topology.

As said, our goal is to find  $\Gamma$ -limits of family of functionals  $(J_h^{2d,\alpha})_{h>0}$  in the strong  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology and of family of functionals  $(J_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  in the strong  $L^4(\omega; \mathbb{R}^3)$  topology. These results have a theoretical value on its own, but they will be used in the Chapter 6 as well. To make results broad enough for both purposes, in this section we will observe functional in both settings regarding Dirichlet boundary conditions: with the Dirichlet boundary conditions on a set  $\gamma_0 \subseteq \partial\omega$  of positive capacity and without the Dirichlet boundary conditions (i.e. for  $\gamma_0 = \emptyset$ ). So in next two results if Dirichlet boundary conditions are not explicitly mentioned in a assumption or claim of the theorem, then it is implied that the claim holds for both mentioned settings ( $|\gamma_0| > 0$  and  $\gamma_0 = \emptyset$ ).

Additionally, even though our model is defined for geometries parametrized by  $\boldsymbol{\theta} \in W^{1,\infty}(\omega; \mathbb{R}^3)$ , for this section we will assume  $\boldsymbol{\theta} \in C^2(\omega; \mathbb{R}^3)$  for technical reasons. Note also that existing results of  $\Gamma$ -convergence for shell related problems starting from 3d elasticity are done for the same increased smoothness of the middle surface geometry.

In the continuation we present two convergence theorems for families  $(J_h^{2d,\alpha})_{h>0}$  and  $(J_{\mathbf{S},h}^{2d,\alpha})_{h>0}$ . Note that in the case of Theorem 5.4.1, apart from the case  $\alpha \in [5/3, 2)$ , we also for now lack a result for the case  $\alpha \in \langle 0, 5/3 \rangle$ . In Theorem 5.4.2 we do have a claim for mentioned case.

**Theorem 5.4.1.** Let the family of functionals  $(J_h^{2d,\alpha})_{h>0}$  be as defined in (5.4.1). Let us assume that conditions N1), N2) and N3) are satisfied.

- a) Let  $\alpha \in \langle -1, 0 \rangle$ . Let additionally  $\gamma_0 = \emptyset$ , i.e. there is no Dirichlet boundary condition. Then the  $\Gamma$ -limit of the family of functionals  $(J_h^{2d,\langle -1,0 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology is

$$J_0^{2d,\langle -1,0 \rangle}(\boldsymbol{\psi}, \mathbf{S}) = - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}'. \quad (5.4.4)$$

- b) Let  $\alpha = 0$ . Then the  $\Gamma$ -limit of the family of functionals  $(J_h^{2d,0})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology is

$$J_0^{2d,0}(\boldsymbol{\psi}, \mathbf{S}) = \begin{cases} \Gamma - \left( \int_{\omega} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}) \sqrt{a} d\mathbf{x}' \right) - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}' & (\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^M, \\ +\infty & \text{otherwise,} \end{cases} \quad (5.4.5)$$

where

$$\mathcal{A}^M := \{(\boldsymbol{\psi}, \mathbf{S}) \in W^{1,4}(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3)) : \boldsymbol{\psi}|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0}, \det \begin{bmatrix} \mathbf{S} \mathbf{a}_3 & \partial_1 \boldsymbol{\psi} & \partial_2 \boldsymbol{\psi} \end{bmatrix} \geq 0 \text{ a.e.} \}. \quad (5.4.6)$$

- c) Let  $\alpha = 2$ . Then the  $\Gamma$ -limit of the family of functionals  $(J_h^{2d,2})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology is

$$J_0^{2d,2}(\boldsymbol{\psi}, \mathbf{S}) = \begin{cases} \frac{1}{12} \int_{\omega} q_f(\mathbf{S}, \nabla' \mathbf{S}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}' & (\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^{\pm F}, \\ +\infty & \text{otherwise,} \end{cases} \quad (5.4.7)$$

where

$$\mathcal{A}^{\pm F} = \{(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^N : \mathbf{S} \mathbf{a}_{\beta} = \partial_{\beta} \boldsymbol{\psi}, \beta = 1, 2 \text{ or } \mathbf{S} \mathbf{a}_{\beta} = -\partial_{\beta} \boldsymbol{\psi}, \beta = 1, 2\}. \quad (5.4.8)$$

If the Dirichlet condition is prescribed on  $\gamma_0$  (with positive capacity), then  $\mathcal{A}^{\pm F} = \mathcal{A}^F$ , where  $\mathcal{A}^F$  is defined in (5.2.2).

- d) Let  $\alpha \in \langle 2, +\infty \rangle$ . Then the  $\Gamma$ -limit of the family of functionals  $(J_h^{2d, \langle 2, +\infty \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology is

$$J_0^{2d, \langle 2, +\infty \rangle}(\psi, \mathbf{S}) = \begin{cases} - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}' & (\psi, \mathbf{S}) \in \mathcal{B}_0^{2d, \langle 2, +\infty \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (5.4.9)$$

where

$$\begin{aligned} \mathcal{B}_0^{2d, \langle 2, +\infty \rangle} &:= \{(\psi, \mathbf{S}) \in C(\omega; \mathbb{R}^3) \times \text{SO}(3) : \psi(\mathbf{x}') = \mathbf{S}\boldsymbol{\theta}(\mathbf{x}')^T + \mathbf{d}, \mathbf{d} \in \mathbb{R}^3 \\ &\text{or } \psi(\mathbf{x}') = -\mathbf{S}\boldsymbol{\theta}(\mathbf{x}')^T + \mathbf{d}, \mathbf{d} \in \mathbb{R}^3\}. \end{aligned} \quad (5.4.10)$$

If the Dirichlet condition is prescribed on  $\gamma_0$  (with positive capacity), then

$$\mathcal{B}_0^{2d, \langle 2, +\infty \rangle} = \{(\boldsymbol{\theta}, \mathbf{I})\}.$$

**Theorem 5.4.2.** Let the family of functionals  $(J_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  be as defined in (5.4.2). Let us assume that conditions **N1**), **N2**) and **N3**) are satisfied.

- a) Let  $\alpha \in \langle -1, 0 \rangle$ . Let additionally  $\gamma_0 = \emptyset$ , i.e. there is no Dirichlet boundary condition. Then the  $\Gamma$ -limit of the family of functionals  $(J_{\mathbf{S},h}^{2d, \langle -1, 0 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^4(\omega; \mathbb{R}^3)$  topology is

$$J_{\mathbf{S},0}^{2d, \langle -1, 0 \rangle}(\psi) = - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}'. \quad (5.4.11)$$

- b) Let  $\alpha = 0$ . Then the  $\Gamma$ -limit of the family of functionals  $(J_{\mathbf{S},h}^{2d,0})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^4(\omega; \mathbb{R}^3)$  topology is

$$J_{\mathbf{S},0}^{2d,0}(\psi) = \begin{cases} \int_{\omega} QW_0(\nabla' \psi) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}' & \psi \in \mathcal{B}_{\mathbf{S},0}^{2d,0}, \\ +\infty & \text{otherwise,} \end{cases} \quad (5.4.12)$$

where

$$\mathcal{B}_{\mathbf{S},0}^{2d,0} := \{\psi \in W^{1,4}(\omega; \mathbb{R}^3) : \psi|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0}\} \quad (5.4.13)$$

and

$$W_0(\mathbf{F}) := \min_{\mathbf{S} \in \text{SO}(3)} w_m(\mathbf{S}, \mathbf{F}). \quad (5.4.14)$$

- c) Let  $\alpha \in \langle 0, 5/3 \rangle$ . Let additionally  $\boldsymbol{\theta}(\mathbf{x}') = (\mathbf{x}', 0)$ ,  $\gamma_0 = \emptyset$  (i.e. no Dirichlet boundary conditions) and  $W_m$  satisfies  $W_m(\mathbf{Y}) \leq C \|\mathbf{Y}\|_F^2$  for some  $C > 0$ . Then the  $\Gamma$ -limit of the family of functionals  $(J_{\mathbf{S},h}^{2d,\langle 0,5/3 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^4(\omega; \mathbb{R}^3)$  topology is

$$J_{\mathbf{S},0}^{2d,\langle 0,5/3 \rangle}(\boldsymbol{\psi}) = \begin{cases} - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} d\mathbf{x}' & \boldsymbol{\psi} \in \mathcal{B}_{\mathbf{S},0}^{2d,\langle 0,5/3 \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (5.4.15)$$

where

$$\mathcal{B}_{\mathbf{S},0}^{2d,\langle 0,5/3 \rangle} := \{\boldsymbol{\psi} \in W^{1,\infty}(\omega; \mathbb{R}^3) : \nabla' \boldsymbol{\psi}^T \nabla' \boldsymbol{\psi} \leq \mathbf{I} \text{ a.e.}\}. \quad (5.4.16)$$

- d) Let  $\alpha = 2$ . Then the  $\Gamma$ -limit of the family of functionals  $(J_{\mathbf{S},h}^{2d,2})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^4(\omega; \mathbb{R}^3)$  topology is

$$J_{\mathbf{S},0}^{2d,2}(\boldsymbol{\psi}) = \begin{cases} \frac{1}{12} \int_{\omega} \tilde{q}_f(\mathbf{b}(\boldsymbol{\psi}) - \mathbf{b}(\boldsymbol{\theta})) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}' & \boldsymbol{\psi} \in \mathcal{B}_{\mathbf{S},0}^{2d,2}, \\ +\infty & \text{otherwise,} \end{cases} \quad (5.4.17)$$

where

$$\mathcal{B}_{\mathbf{S},0}^{2d,2} := \{\boldsymbol{\psi} \in W^{2,2}(\omega; \mathbb{R}^3) : \boldsymbol{\psi}|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0}, \nabla \boldsymbol{\psi}^T \nabla \boldsymbol{\psi} = \nabla \boldsymbol{\theta}^T \nabla \boldsymbol{\theta}\}, \quad (5.4.18)$$

$\mathbf{b}(\boldsymbol{\psi}) = (b_{\beta_1, \beta_2}(\boldsymbol{\psi}))_{\beta_1, \beta_2}$  is the curvature tensor and  $\tilde{q}_f$  is defined in (5.3.8).

- e) Let  $\alpha \in \langle 2, +\infty \rangle$ . Then the  $\Gamma$ -limit of family of functionals  $(J_{\mathbf{S},h}^{2d,\langle 2,+\infty \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^4(\omega; \mathbb{R}^3)$  topology is

$$J_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle}(\boldsymbol{\psi}) = \begin{cases} - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}' & \boldsymbol{\psi} \in \mathcal{B}_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (5.4.19)$$

where

$$\begin{aligned} \mathcal{B}_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle} &:= \{\boldsymbol{\psi} \in C(\omega; \mathbb{R}^3) : \\ &\exists \mathbf{R} \in \text{SO}(3), \mathbf{d} \in \mathbb{R}^3 \text{ s.t. } \boldsymbol{\psi}(\mathbf{x}') = \mathbf{R}\boldsymbol{\theta}(\mathbf{x}')^T + \mathbf{d}\}. \end{aligned} \quad (5.4.20)$$

If the Dirichlet condition is prescribed on  $\gamma_0$  (with positive capacity), then

$$\mathcal{B}_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle} = \{\boldsymbol{\theta}\}.$$



Proofs from Theorem 5.4.1.b) and Theorem 5.4.2.b) are in Subsection 5.4.2. Claims from Theorem 5.4.1.c) and Theorem 5.4.2.d) will be proved in Subsection 5.4.4. These results in the case for  $\alpha = 0$  and  $\alpha = 2$  are already presented in [61]. Claim from Theorem 5.4.2.c) will be proved in Subsection 5.4.5. The four remaining claims from last two theorems will be proved in Subsection 5.4.6.

### 5.4.1. Technical lemmas

We start asymptotical analysis with four technical results related to coercivity and growth properties of terms in the energy functional, and to certain limits of terms in the membrane and flexural strains that will be needed later on.

**Lemma 5.4.3.** a) Let  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$ . Then the functional

$$m(\boldsymbol{\psi}) := \int_{\omega} W_m \left( \begin{bmatrix} (\mathbf{S}^T \nabla' \boldsymbol{\psi} + \nabla' \boldsymbol{\theta})^T (\mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}) \\ (\mathbf{S} \mathbf{a}_3)^T \nabla' \boldsymbol{\psi} \end{bmatrix} \right) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}'$$

is coercive on  $W^{1,4}(\omega; \mathbb{R}^3)$ , i.e. there are constants  $c_a > 0$  and  $C_a \in \mathbb{R}$  such that for all  $\boldsymbol{\psi} \in W^{1,4}(\omega; \mathbb{R}^3)$  one has

$$\begin{aligned} \|\boldsymbol{\psi}\|_{L^4(\omega; \mathbb{R}^3)} + m(\boldsymbol{\psi}) &\geq c_a \left( \|\nabla' \boldsymbol{\psi}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \right. \\ &\quad + \|\nabla' \boldsymbol{\theta}^T \mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\psi}^T \mathbf{S} \nabla' \boldsymbol{\theta}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \\ &\quad \left. + \|(\mathbf{S} \mathbf{a}_3)^T \nabla' \boldsymbol{\psi}\|_{L^2(\omega; \mathbb{R}^2)}^2 + \|\boldsymbol{\psi}\|_{W^{1,4}(\omega; \mathbb{R}^3)}^4 \right) - C_a, \end{aligned}$$

where  $C_a$  is the undeformed geometry dependent.

b) The functional

$$f(\mathbf{S}) := \int_{\omega} Q_f(\mathbf{S}^T \nabla' \mathbf{S}) \sqrt{a} d\mathbf{x}'$$

is coercive on  $W^{1,2}(\omega; \text{SO}(3))$ , i.e., there is a constant  $c_b > 0$  such that for all  $\mathbf{S} \in W^{1,2}(\omega; \text{SO}(3))$  one has

$$\|\mathbf{S}\|_{L^2(\omega; \text{SO}(3))} + f(\mathbf{S}) \geq c_b \|\mathbf{S}\|_{W^{1,2}(\omega; \text{SO}(3))}.$$

*Proof.* Let  $M := \|\mathbf{f}\|_{L^{4/3}(\omega; \mathbb{R}^3)}$ . Applying coercivity of  $W_m$ , and then the orthogonality

mentioned in (5.3.2), we obtain

$$\begin{aligned}
m(\psi) &\geq c_1 \left( \| (S^T \nabla' \psi + \nabla' \theta)^T (S^T \nabla' \psi - \nabla' \theta) \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \right. \\
&\quad \left. + \| (S a_3)^T \nabla' \psi \|_{L^2(\omega; \mathbb{R}^2)}^2 \right) - M \|\psi\|_{L^4(\omega; \mathbb{R}^3)} \\
&\geq c_1 \left( \| \nabla' \psi^T \nabla' \psi - \nabla' \theta^T \nabla' \theta \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + \| \nabla' \theta^T S^T \nabla' \psi - \nabla' \psi^T S \nabla' \theta \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \right. \\
&\quad \left. + \| (S a_3)^T \nabla' \psi \|_{L^2(\omega; \mathbb{R}^2)}^2 \right) - M \|\psi\|_{L^4(\omega; \mathbb{R}^3)}.
\end{aligned}$$

Next we use the property  $\|X\|_2^2 = \|X^T X\|_2$  for general matrix  $X$ , equivalence of all matrix norms and the triangle inequality to prove

$$\begin{aligned}
\| \nabla' \psi \|_{L^4(\omega; \mathbb{R}^{3 \times 2})}^4 &\leq \| \nabla' \psi^T \nabla' \psi \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \\
&\leq 2 \| \nabla' \psi^T \nabla' \psi - \nabla' \theta^T \nabla' \theta \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + 2 \| \nabla' \theta^T \nabla' \theta \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2.
\end{aligned}$$

By adding the contribution of  $\|\psi\|_{L^4(\omega; \mathbb{R}^3)}^4$ , we have

$$\begin{aligned}
\|\psi\|_{L^4(\omega; \mathbb{R}^3)}^4 + m(\psi) &\geq c'_1 \left( \| \nabla' \psi^T \nabla' \psi - \nabla' \theta^T \nabla' \theta \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \right. \\
&\quad + \| \nabla' \theta^T S^T \nabla' \psi - \nabla' \psi^T S \nabla' \theta \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \\
&\quad + \| (S a_3)^T \nabla' \psi \|_{L^2(\omega; \mathbb{R}^2)}^2 + \|\psi\|_{W^{1,4}(\omega; \mathbb{R}^3)}^4 \Big) \\
&\quad - M \|\psi\|_{L^4(\omega; \mathbb{R}^3)} - C \\
&\geq c'_1 \left( \| \nabla' \psi^T \nabla' \psi - \nabla' \theta^T \nabla' \theta \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \right. \\
&\quad + \| \nabla' \theta^T S^T \nabla' \psi - \nabla' \psi^T S \nabla' \theta \|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \\
&\quad + \| (S a_3)^T \nabla' \psi \|_{L^2(\omega; \mathbb{R}^2)}^2 + \|\psi\|_{W^{1,4}(\omega; \mathbb{R}^3)}^4 \Big) - C,
\end{aligned}$$

since the polynomial  $x \mapsto ax^4 + bx$ ,  $a > 0$ , is bounded from below by some constant.

To prove b) we just apply the coercivity of  $Q_f$ , the fact that  $S$  is with values in  $SO(3)$  and the Poincaré inequality.  $\square$

**Lemma 5.4.4.** a) There is a constant  $M_a > 0$  such that for all  $S \in L^2(\omega; SO(3))$  and  $\psi \in W^{1,4}(\omega; \mathbb{R}^3)$  one has

$$\int_{\omega} w_m(S, \nabla' \psi) \sqrt{a} d\mathbf{x}' \leq M_a (1 + \|\nabla \psi\|_{L^4(\omega; \mathbb{R}^{3 \times 2})}^4),$$

where  $M_a$  is undeformed geometry dependent.

b) There is a constant  $M_b > 0$  such that for all  $S \in W^{1,2}(\omega; SO(3))$  one has

$$\int_{\omega} q_f(S, \nabla' S) \sqrt{a} d\mathbf{x}' \leq M_b (1 + \|\nabla S\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}^2),$$

where  $M_b$  is undeformed geometry dependent.

*Proof.* For the a) part, from the growth condition N2) and by similar calculations as in Lemma 5.4.3 we have

$$\begin{aligned}
& \int_{\omega} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}) \sqrt{a} d\mathbf{x}' \\
& \leq M_{a,1} \left( \|\nabla' \boldsymbol{\psi}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \right. \\
& \quad \left. + \|\nabla' \boldsymbol{\theta}^T \mathbf{S}^T \nabla' \boldsymbol{\psi} - \nabla' \boldsymbol{\psi}^T \mathbf{S} \nabla' \boldsymbol{\theta}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + \|(\mathbf{S} \mathbf{a}_3)^T \nabla' \boldsymbol{\psi}\|_{L^2(\omega; \mathbb{R}^2)}^2 + C \right) \\
& \leq M_{a,2} \left( \|\nabla' \boldsymbol{\psi}^T \nabla' \boldsymbol{\psi}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + c \|\nabla' \boldsymbol{\psi}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + C \right) \\
& \leq M_{a,3} (1 + \|\nabla \boldsymbol{\psi}\|_{L^4(\omega; \mathbb{R}^{3 \times 2})}^4),
\end{aligned} \tag{5.4.21}$$

where we again used the identity  $\|\mathbf{X}\|_2^2 = \|\mathbf{X}^T \mathbf{X}\|_2$ . For the b) part we have similar and even simpler calculations.  $\square$

**Lemma 5.4.5.** Let  $(\boldsymbol{\psi}_n, \mathbf{S}_n)_{n \geq 1} \subset \mathcal{A}^N$ ,  $(\boldsymbol{\psi}_0, \mathbf{S}_0) \in \mathcal{A}^N$  be such that

$$(\boldsymbol{\psi}_n, \mathbf{S}_n) \rightharpoonup (\boldsymbol{\psi}_0, \mathbf{S}_0) \text{ in } W^{1,4}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3)). \tag{5.4.22}$$

Then

$$\nabla' \boldsymbol{\psi}_n^T \mathbf{S}_n \nabla' \boldsymbol{\theta} - \nabla' \boldsymbol{\theta}^T \mathbf{S}_n^T \nabla' \boldsymbol{\psi}_n \rightharpoonup \nabla' \boldsymbol{\psi}^T \mathbf{S} \nabla' \boldsymbol{\theta} - \nabla' \boldsymbol{\theta}^T \mathbf{S}^T \nabla' \boldsymbol{\psi} \text{ in } L^4(\omega; \mathbb{R}^{3 \times 2}), \tag{5.4.23}$$

$$\mathbf{S}_n^T \nabla' \mathbf{S}_n \rightharpoonup \mathbf{S}_0^T \nabla' \mathbf{S}_0 \text{ in } L^2(\omega; \mathbb{R}^{3 \times 2}). \tag{5.4.24}$$

$$\nabla' \boldsymbol{\psi}_n^T \mathbf{S}_n \mathbf{a}_3 \rightharpoonup \nabla' \boldsymbol{\psi}_0^T \mathbf{S}_0 \mathbf{a}_3 \text{ in } L^4(\omega; \mathbb{R}^2). \tag{5.4.25}$$

*Proof.* Since the measurable functions with values in  $\text{SO}(3)$  belong to  $L^\infty(\omega; \mathbb{R}^{3 \times 3})$ , the sequence  $(\nabla' \boldsymbol{\psi}_n^T \mathbf{S}_n \nabla' \boldsymbol{\theta} - \nabla' \boldsymbol{\theta}^T \mathbf{S}_n^T \nabla' \boldsymbol{\psi}_n)_{n \geq 1}$  is bounded in  $L^4(\omega; \mathbb{R}^{3 \times 2})$ , so it has weakly convergent subsequence  $(\nabla' \boldsymbol{\psi}_{n_k}^T \mathbf{S}_{n_k} \nabla' \boldsymbol{\theta} - \nabla' \boldsymbol{\theta}^T \mathbf{S}_{n_k}^T \nabla' \boldsymbol{\psi}_{n_k})_{n_k \geq 1}$  in  $L^4(\omega; \mathbb{R}^{3 \times 2})$ .

On the other hand, from compact embedding of  $H^1(\omega)$  to  $L^2(\omega)$ , we have that  $\mathbf{S}_{n_k} \rightarrow \mathbf{S}_0$  in  $L^2(\omega; \mathbb{R}^{3 \times 2})$ . Thus

$$(\nabla' \boldsymbol{\psi}_{n_k}^T \mathbf{S}_{n_k} \nabla' \boldsymbol{\theta} - \nabla' \boldsymbol{\theta}^T \mathbf{S}_{n_k}^T \nabla' \boldsymbol{\psi}_{n_k}) \rightharpoonup (\nabla' \boldsymbol{\psi}^T \mathbf{S} \nabla' \boldsymbol{\theta} - \nabla' \boldsymbol{\theta}^T \mathbf{S}^T \nabla' \boldsymbol{\psi}) \text{ in } L^1(\omega; \mathbb{R}^{3 \times 2}). \tag{5.4.26}$$

Uniqueness of limits in distributions now implies the convergence (5.4.23).

The proof is analogous for sequences in (5.4.24) and (5.4.25).  $\square$

**Remark 5.4.6.** We will use similar result without mention: if  $(\mathbf{S}_n)_{n \geq 1} \subset L^2(\omega; \text{SO}(3))$  converge strongly to  $\mathbf{S} \in L^2(\omega; \mathbb{R}^{3 \times 3})$ , then  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$ .

Namely, we firstly have that  $(\mathbf{S}_n^T \mathbf{S}_n)_{n \geq 1}$  converges in  $L^1(\omega; \mathbb{R}^{3 \times 3})$ , so  $\mathbf{S}$  is unitary matrix. Secondly, the determinant of a  $3 \times 3$  matrix can be observed as a polynomial of the degree 3 of its elements. Since  $\mathbf{S}_n \in \text{SO}(3)$ , specially  $\mathbf{S} \in L^\infty(\omega; \mathbb{R}^{3 \times 3})$  so there is a weak\*  $L^\infty(\omega; \mathbb{R}^{3 \times 3})$  convergent subsequence  $(\mathbf{S}_{n_k})_{k \geq 1}$ , converging again to  $\mathbf{S}$  due to the uniqueness of limits. In each addend  $s_{i_1, j_1}^{n_k} s_{i_2, j_2}^{n_k} s_{i_3, j_3}^{n_k}$  in  $\det(\mathbf{S}_{n_k})$  we use strong  $L^2(\omega; \mathbb{R}^{3 \times 3})$  convergence for first two factors and weak\*  $L^\infty(\omega; \mathbb{R}^{3 \times 3})$  convergence for the third, to obtain that  $(\det(\mathbf{S}_{n_k}))_{k \geq 1}$  converges weakly in  $L^1(\omega; \mathbb{R}^{3 \times 3})$  to  $\det \mathbf{S}$ . Since  $1 = \det(\mathbf{S}_{n_k}) \rightarrow \det \mathbf{S}$ , we have  $\det \mathbf{S} = 1$ , so  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$ .

#### 5.4.2. Membrane shell model - convergence for the case $\alpha = 0$

In this subsection we prove the claims of Theorem 5.4.1.b) and Theorem 5.4.2.b). The goal is to obtain a limit and compare the result with the membrane shell model obtained as the  $\Gamma$ -limit of the three-dimensional elastic energy in [51]. The result in [51] is a generalization of the result for the planar middle surface geometry presented in Theorem 3.2.1.b) and [52].

We define  $\tilde{J}_0^{2d,0}$  as an accumulation point (in the sense of  $\Gamma$ -convergence) of the family of functionals  $(J_h^{2d,0})_{h>0}$ . In Theorem 5.4.1.b) we identify the  $\Gamma$ -limit of family of functionals  $(J_h^{2d,0})_{h>0}$  and prove that it is independent of the flexural energy and derivatives of the function  $\mathbf{S}$ . Thus the matrix of rotations  $\mathbf{S}$  of the cross-section of the shell appears in the limit just as a parameter (without measuring its deformation). In order to do that we consider the following function set

$$\mathcal{A}^M := \{(\boldsymbol{\psi}, \mathbf{S}) \in W^{1,4}(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3)) : \boldsymbol{\psi}|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0}, \det \begin{bmatrix} \mathbf{S} \mathbf{a}_3 & \partial_1 \boldsymbol{\psi} & \partial_2 \boldsymbol{\psi} \end{bmatrix} \geq 0 \text{ a.e.}\}, \quad (5.4.27)$$

and the functional

$$\tilde{J}_m(\boldsymbol{\psi}, \mathbf{S}) = \begin{cases} \int_{\omega} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}) \sqrt{a} dx' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} dx' & (\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^M, \\ +\infty & \text{otherwise.} \end{cases} \quad (5.4.28)$$

We will actually prove that  $\tilde{J}_0^{2d,0}$  is weakly lower semicontinuous envelope of  $\tilde{J}_m$ . The main obstacle then is elimination of derivatives of  $\mathbf{S}$  in the limit set of admissible functions (note that due to lack of the regularity of  $\mathbf{S}$  we even lose boundary conditions for rotations in the set  $\mathcal{A}^M$ ). It is achieved in Lemma A.1, technical and complicated density result of  $\mathcal{A}^N$  in  $\mathcal{A}^M$ . We are able to prove this density property only for  $C^2$  parametrizations of the undeformed middle shell given on polygonal domains, i.e.,  $\boldsymbol{\theta} \in C^2(\bar{\omega})$ , and only in two cases: the Dirichlet boundary condition is applied on the entire boundary ( $\gamma_0 = \emptyset$ ) or the parametrization of undeformed domain is piecewise affine on the part of the domain where the Dirichlet boundary condition is prescribed (including the case  $\gamma_0 = \emptyset$ ). However, we feel the statement holds in a more general setting.

*Proof of Theorem 5.4.1.b).* If we prove that an accumulation point  $\tilde{J}_0^{2d,0}$  satisfies the identity  $\tilde{J}_0^{2d,0}(\boldsymbol{\psi}, \mathbf{S}) = \Gamma - \tilde{J}_m(\boldsymbol{\psi}, \mathbf{S})$ , the proof will be over: since sequentially weakly lower semicontinuous envelope  $\Gamma - \tilde{J}_m(\boldsymbol{\psi}, \mathbf{S})$  is unique, we can conclude that  $\tilde{J}_0^{2d,0}(\boldsymbol{\psi}, \mathbf{S})$  is unique as well and thus the whole family  $(J_h^{2d,0}(\boldsymbol{\psi}, \mathbf{S}))_{h>0}$   $\Gamma$ -converges to the same functional.

We will prove inequalities  $\tilde{J}_0^{2d,0}(\boldsymbol{\psi}, \mathbf{S}) \leq \tilde{J}_m(\boldsymbol{\psi}, \mathbf{S}) \leq J_h^{2d,0}(\boldsymbol{\psi}, \mathbf{S})$ . The second one is clear. After we prove the first one, we can take the  $\Gamma$ -limit in the inequality and obtain the statement.

Let us take arbitrary  $(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^M$ . By Lemma A.1 there exists  $(\boldsymbol{\psi}_n, \mathbf{S}_n)_{n \geq 1} \subset \mathcal{A}^N$  such that  $(\boldsymbol{\psi}_n, \mathbf{S}_n) \rightarrow (\boldsymbol{\psi}, \mathbf{S})$  in  $W^{1,4}(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  strongly. The elements of this sequence we can plug into  $J_h^{2d,0}(\boldsymbol{\psi}, \mathbf{S})$  to obtain the estimate. Let us define  $(\boldsymbol{\psi}_h, \mathbf{S}_h)_{h>0} \subset \mathcal{A}^N$  as  $(\boldsymbol{\psi}_h, \mathbf{S}_h) = (\boldsymbol{\psi}_{n(h)}, \mathbf{S}_{n(h)})$  where

$$n(h) := \arg \max_{n \geq 1} \left\{ \|\nabla' \mathbf{S}_n\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}^2 \leq 1/h \right\} \quad (5.4.29)$$

in case that  $\|\nabla' \mathbf{S}_n\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}$  is not uniformly bounded (note that  $1/h \rightarrow +\infty$  as  $h \rightarrow 0$ , so the mapping is well defined for all sufficiently small values of  $h$ ), and  $n(h) = \lceil 1/h \rceil$  if  $\|\nabla' \mathbf{S}_n\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}$  is uniformly bounded.

If  $\|\nabla' \mathbf{S}_n\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}$  is uniformly bounded, then clearly  $n(h) \rightarrow +\infty$  and due to  $1/h \rightarrow +\infty$  the bound  $\|\nabla' \mathbf{S}_n\|_{L^2(\omega; \mathbb{R}^{3 \times 6})} < 1/h$  is satisfied for sufficiently small values of  $h$ . If  $\|\nabla' \mathbf{S}_n\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}$  is not uniformly bounded, as  $h \rightarrow 0$ , due to  $1/h \rightarrow +\infty$  we have  $n(h) \rightarrow +\infty$ . In both cases we have that  $(\boldsymbol{\psi}_h, \mathbf{S}_h) = (\boldsymbol{\psi}_{n(h)}, \mathbf{S}_{n(h)})$  converges

to  $(\boldsymbol{\psi}, \mathbf{S})$  in  $W^{1,4}(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  strongly. However, due to the definition of the sequence, for sufficiently small  $h$ :

$$\frac{h^2}{12} \int_{\omega} q_f(\mathbf{S}, \nabla' \mathbf{S}) \leq Ch \quad (5.4.30)$$

and by the Lebesgue dominated convergence theorem (using also Lemma 5.4.4)

$$\int_{\omega} w_m(\mathbf{S}_h, \nabla' \boldsymbol{\psi}_h) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi}_h \sqrt{a} d\mathbf{x}' \rightarrow \int_{\omega} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}'. \quad (5.4.31)$$

Thus, for  $h$  small enough we have

$$J_h^{2d,0}(\boldsymbol{\psi}_h, \mathbf{S}_h) \leq \tilde{J}_m(\boldsymbol{\psi}_h, \mathbf{S}_h) + Ch, \quad (5.4.32)$$

and, after  $h \rightarrow 0$ ,

$$\tilde{J}_0^{2d,0}(\boldsymbol{\psi}, \mathbf{S}) \leq \lim_{h \rightarrow 0} J_h^{2d,0}(\boldsymbol{\psi}_h, \mathbf{S}_h) \leq \tilde{J}_m(\boldsymbol{\psi}, \mathbf{S}), \quad (5.4.33)$$

which completes the proof.  $\square$

As announced in the introduction of this section, we can also observe the  $\Gamma$ -limit of functionals  $J_{\mathbf{S},h}^{2d,0}(\boldsymbol{\psi}) = \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} J_h^{2d,0}(\boldsymbol{\psi}, \mathbf{S})$ . Let  $\tilde{J}_{\mathbf{S},0}^{2d,0}(\boldsymbol{\psi})$  denotes a  $\Gamma$ -accumulation point of the family  $(J_{\mathbf{S},h}^{2d,0}(\boldsymbol{\psi}))_{h>0}$  in the strong  $L^4(\omega; \mathbb{R}^3)$  topology.

**Proposition 5.4.7.** It holds

$$\inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} \int_{\omega} w_m(\mathbf{S}(\mathbf{x}'), \nabla' \boldsymbol{\psi}(\mathbf{x}')) \sqrt{a} d\mathbf{x}' = \int_{\omega} \min_{\mathbf{S} \in \text{SO}(3)} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}(\mathbf{x}')) \sqrt{a} d\mathbf{x}' \quad (5.4.34)$$

*Proof.* Let us denote the left hand side and right hand side of (5.4.34) with  $I_1(\boldsymbol{\psi})$  and  $I_2(\boldsymbol{\psi})$ , respectively. We need to prove  $I_1(\boldsymbol{\psi}) \leq I_2(\boldsymbol{\psi})$ , since the other inequality is a consequence of taking infimum over larger set.

Let us fix  $\boldsymbol{\psi} \in W^{1,4}(\omega; \mathbb{R}^3)$ . Let us define function  $g : \omega \times \text{SO}(3) \rightarrow \overline{\mathbb{R}}$ ,

$$g(\mathbf{x}', \mathbf{S}) = \begin{cases} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}(\mathbf{x}')) & \det \begin{bmatrix} \mathbf{S} \mathbf{a}_3(\mathbf{x}') & \partial_1 \boldsymbol{\psi}(\mathbf{x}') & \partial_2 \boldsymbol{\psi}(\mathbf{x}') \end{bmatrix} \geq 0, \\ +\infty & \text{otherwise.} \end{cases} \quad (5.4.35)$$

This function is a normal integrand (when it is not equal to  $+\infty$ , it is continuous in the second variable and measurable in the first, and the domain when it is not equal to  $+\infty$

is closed). Also, note that for each  $\psi$  and each  $\mathbf{x}'$ , the function  $g(\mathbf{x}', \cdot)$  is not identically equal to  $+\infty$ .

We can apply the Measurable selection lemma ([34], Chapter 8, Theorem 1.2) to this function and conclude that there exists measurable function  $\mathbf{R}_\psi : \omega \rightarrow \text{SO}(3)$  such that

$$w_m(\mathbf{R}_\psi(\mathbf{x}'), \nabla' \psi(\mathbf{x}')) = \inf_{\mathbf{S} \in \text{SO}(3)} w_m(\mathbf{S}, \nabla' \psi(\mathbf{x}')). \quad (5.4.36)$$

Since  $\mathbf{R}_\psi$  is measurable and rotation, it is trivially in  $L^2(\omega; \text{SO}(3))$ . Now, it is clear that

$$\begin{aligned} I_1(\psi) &= \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} \int_{\omega} w_m(\mathbf{S}(\mathbf{x}'), \nabla' \psi(\mathbf{x}')) \sqrt{a} d\mathbf{x}' \\ &\leq \int_{\omega} w_m(\mathbf{R}_\psi(\mathbf{x}'), \nabla' \psi(\mathbf{x}')) \sqrt{a} d\mathbf{x}' = I_2(\psi). \end{aligned} \quad (5.4.37)$$

which completes the proof.  $\square$

*Proof of Theorem 5.4.2.b).* We will actually prove equalities

$$\tilde{J}_{\mathbf{S},0}^{2d,0}(\psi) = \Gamma - \left( \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} J_0^{2d,0}(\psi, \mathbf{S}) \right) = \int_{\omega} QW_0(\nabla' \psi) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}', \quad (5.4.38)$$

where

$$W_0(\mathbf{F}) := \min_{\mathbf{S} \in \text{SO}(3)} w_m(\mathbf{S}, \mathbf{F}), \quad (5.4.39)$$

and  $Q$  denotes taking its quasiconvex envelope. The claim of Theorem 5.4.2.b) will be a consequence of the unique value of the accumulation point.

In the inequality for  $(\psi, \mathbf{S}) \in \mathcal{A}^N$

$$\begin{aligned} J_h^{2d,0}(\psi, \mathbf{S}) &= \int_{\omega} w_m(\mathbf{S}, \nabla' \psi) \sqrt{a} d\mathbf{x}' + \frac{h^2}{12} \int_{\omega} q_f(\mathbf{S}, \nabla' \mathbf{S}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}' \\ &\geq \int_{\omega} w_m(\mathbf{S}, \nabla' \psi) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}', \end{aligned}$$

we firstly take  $\inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))}$ , to obtain

$$\begin{aligned} J_{\mathbf{S},h}^{2d,0}(\psi) &\geq \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} \int_{\omega} w_m(\mathbf{S}, \nabla' \psi) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}' \\ &= \int_{\omega} \inf_{\mathbf{S} \in \text{SO}(3)} w_m(\mathbf{S}, \nabla' \psi(\mathbf{x}')) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}', \end{aligned}$$

and then take  $\Gamma$ -limit of both sides of inequality, to obtain

$$\tilde{J}_{\mathbf{S},0}^{2d,0}(\psi) \geq \int_{\omega} QW_0(\nabla' \psi) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}'. \quad (5.4.40)$$

On the other hand, for all  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$  we have  $J_{\mathbf{S},h}^{2d,0}(\boldsymbol{\psi}) \leq J_h^{2d,0}(\boldsymbol{\psi}, \mathbf{S})$ . After taking  $\Gamma$ -limit and by using Theorem 5.4.1.b), we obtain

$$\tilde{J}_{\mathbf{S},0}^{2d,0}(\boldsymbol{\psi}) \leq J_0^{2d,0}(\boldsymbol{\psi}, \mathbf{S}) = \Gamma - \tilde{J}_m(\boldsymbol{\psi}, \mathbf{S}) \leq \tilde{J}_m(\boldsymbol{\psi}, \mathbf{S}).$$

After taking  $\inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))}$  we obtain

$$\begin{aligned} \tilde{J}_{\mathbf{S},0}^{2d,0}(\boldsymbol{\psi}) &\leq \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} J_0^{2d,0}(\boldsymbol{\psi}, \mathbf{S}) \\ &\leq \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} \int_{\omega} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}'. \end{aligned}$$

Then we use Proposition 5.4.7 and take  $\Gamma$ -limit of both sides of inequality to obtain

$$\tilde{J}_{\mathbf{S},0}^{2d,0}(\boldsymbol{\psi}) \leq \Gamma - \left( \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} J_0^{2d,0}(\boldsymbol{\psi}, \mathbf{S}) \right) \leq \int_{\omega} QW_0(\nabla' \boldsymbol{\psi}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}'. \quad (5.4.41)$$

Inequalities (5.4.40) and (5.4.41) complete the proof.  $\square$

**Remark 5.4.8.** Note that we have obtained the following

$$\begin{aligned} &\Gamma \left( \left( \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} J_h^{2d,0}(\boldsymbol{\psi}, \mathbf{S}) \right)_{h>0} \right) \\ &= \Gamma \left( \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} \Gamma(J_h^{2d,0}(\boldsymbol{\psi}, \mathbf{S})) \right) \\ &= \Gamma - \left( \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} \Gamma(\tilde{J}_m(\boldsymbol{\psi}, \mathbf{S})) \right) \\ &= \Gamma - \left( \int_{\omega} \min_{\mathbf{S} \in \text{SO}(3)} w_m(\mathbf{S}, \nabla' \boldsymbol{\psi}) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}' \right). \end{aligned}$$

The first equation implies that two limiting procedures, taking  $\Gamma$ -limit and taking infimum over  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$ , commute. The second equation implies that the flexural energy in the limit can be neglected, while the third equation implies that the same limit can be obtained by taking pointwise infimum with respect to  $\mathbf{S}$  in the energy density function and then the  $\Gamma$ -limit (sequentially weakly lower semicontinuous envelope).

### 5.4.3. Membrane shell model – further properties

In this subsection we explore further properties of the limit membrane model, related to the short maps and in the particular case of the St. Venant–Kirchhoff material.



Let us consider deformation  $\tilde{\psi} : \boldsymbol{\theta}(\omega) \rightarrow \mathbb{R}^3$  on a referent configuration  $\boldsymbol{\theta}(\omega)$ . Such map is called short map (or nonextending) if  $\nabla' \tilde{\psi}^T \nabla' \tilde{\psi} \leq \mathbf{I}$ . This is equivalent to the condition that all singular values of  $\nabla' \tilde{\psi}$  are less than 1. We first characterize short maps in terms of the function  $\psi = \tilde{\psi} \circ \boldsymbol{\theta}$  using the following lemma.

**Lemma 5.4.9.** The singular values of  $(\nabla' \tilde{\psi}) \circ \boldsymbol{\theta}$  are the same as singular values of  $\nabla' \psi \sqrt{\mathbf{A}^c}$ .

*Proof.* From  $\psi = \tilde{\psi} \circ \boldsymbol{\theta}$  we have

$$\nabla' \psi = \nabla' \tilde{\psi} \circ \boldsymbol{\theta} \nabla' \boldsymbol{\theta} = \nabla' \tilde{\psi} \circ \boldsymbol{\theta} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}.$$

A simple calculation then shows that

$$\nabla' \psi \begin{bmatrix} (\mathbf{a}^1)^T \\ (\mathbf{a}^2)^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \nabla' \psi = \nabla' \tilde{\psi} \circ \boldsymbol{\theta} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$$

and thus

$$\nabla' \psi \begin{bmatrix} (\mathbf{a}^1)^T \\ (\mathbf{a}^2)^T \end{bmatrix} = \nabla' \tilde{\psi} \circ \boldsymbol{\theta}. \quad (5.4.42)$$

On the other hand, singular values of the matrix  $\nabla' \psi \begin{bmatrix} (\mathbf{a}^1)^T \\ (\mathbf{a}^2)^T \end{bmatrix}$  are square root of two largest eigenvalues of the matrix

$$\begin{aligned} & \left( \nabla' \psi \begin{bmatrix} (\mathbf{a}^1)^T \\ (\mathbf{a}^2)^T \end{bmatrix} \right) \left( \nabla' \psi \begin{bmatrix} (\mathbf{a}^1)^T \\ (\mathbf{a}^2)^T \end{bmatrix} \right)^T \\ &= \nabla' \psi \begin{bmatrix} (\mathbf{a}^1)^T \\ (\mathbf{a}^2)^T \end{bmatrix} \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 \end{bmatrix} \nabla' \psi^T = \nabla' \psi \mathbf{A}^c \nabla' \psi^T \\ &= \nabla' \psi \sqrt{\mathbf{A}^c} \left( \nabla' \psi \sqrt{\mathbf{A}^c} \right)^T. \end{aligned}$$

Thus, matrices  $\nabla' \psi \begin{bmatrix} (\mathbf{a}^1)^T \\ (\mathbf{a}^2)^T \end{bmatrix} = \nabla' \tilde{\psi} \circ \boldsymbol{\theta}$  and  $\nabla' \psi \sqrt{\mathbf{A}^c}$  have the same singular values. □

As a consequence we have that  $\psi$  is a short map if and only if singular values of  $\nabla' \psi \sqrt{\mathbf{A}^c}$  are less or equal 1.

In the next proposition we prove the property of the energy density function  $QW_0$  of the limit membrane model, obtained as the limit of the problem (5.2.4) for energy density functions given in (5.2.5), that it is zero on short maps. The same property is obtained for the nonlinear membrane model in [52] and nonlinear membrane shell model in [51] derived from three-dimensional elasticity. The additional assumption is that  $(\mathbf{I}, \nabla' \boldsymbol{\theta})$  is the global minima of the energy which corresponds to  $\boldsymbol{\theta}(\bar{\omega})$  being natural state of the shell.

**Proposition 5.4.10.** Let  $w_m$  be such that  $w_m(\mathbf{I}, \nabla' \boldsymbol{\theta}) = 0$ . Then  $QW_0(\mathbf{F}) = 0$ , for all  $\mathbf{F}$  such that  $\sigma_i(\mathbf{F}\sqrt{\mathbf{A}^c}) \leq 1$ ,  $i = 1, 2$ , where  $\sigma_i(\mathbf{F})$  denotes  $i^{th}$  largest singular value of the matrix  $\mathbf{F}$ .

*Proof.* From N3) we have  $w_m \geq 0$ . From this and since  $w_m(\mathbf{I}, \nabla' \boldsymbol{\theta}) = 0$  (i.e. the residual stress is zero), we have  $W_0(\nabla' \boldsymbol{\theta}) = 0$  and  $W_0(\mathbf{F}) \geq 0$ , and  $W_0(\mathbf{SF}) = W_0(\mathbf{F})$  (due to the infimum in the definition of  $W_0$  and the assumed structure of the strain).

Let us introduce the function  $Y_0$  such that

$$Y_0(\mathbf{F}\sqrt{\mathbf{A}^c}) = W_0(\mathbf{F}). \quad (5.4.43)$$

We will prove that  $QY_0(\mathbf{F}) = 0$  whenever  $\sigma_i(\mathbf{F}) \leq 1$ ,  $i = 1, 2$ .

Function  $Y_0$ . satisfies:  $Y_0(\mathbf{F}) \geq 0$  and  $Y_0(\mathbf{SF}) = Y_0(\mathbf{F})$ . Also, since for matrix  $\mathbf{B} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \sqrt{\mathbf{A}^c} = \nabla' \boldsymbol{\theta} \sqrt{\mathbf{A}^c}$  we have

$$\left( \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \sqrt{\mathbf{A}^c} \right)^T \left( \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \sqrt{\mathbf{A}^c} \right) = \sqrt{\mathbf{A}^c} \mathbf{A}_c \sqrt{\mathbf{A}^c} = \mathbf{I}_2, \quad (5.4.44)$$

there exists  $\bar{\mathbf{B}} \in \text{SO}(3)$  such that  $\bar{\mathbf{B}} \mathbf{e}_\beta = \mathbf{B} \mathbf{e}_\beta$ . Then

$$0 = W_0(\nabla' \boldsymbol{\theta}) = Y_0(\mathbf{B}) = Y_0(\bar{\mathbf{B}}^T \mathbf{B}) = Y_0 \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right). \quad (5.4.45)$$

Now we proceed by applying the same technique as in Theorem 10 from [51] to prove that  $QY_0(\mathbf{F}) = 0$ , whenever  $\sigma_i(\mathbf{F}) \leq 1$ ,  $i = 1, 2$ . Due to the QR decomposition and the property  $Y_0(\mathbf{SF}) = Y_0(\mathbf{F})$ , it is sufficient to prove the claim only for matrices  $\mathbf{F}$  that satisfy  $\mathbf{e}_3^T \mathbf{F} = 0$ . So, without change of notation from now on we observe  $Y_0$  as defined

on  $2 \times 2$  matrices  $\mathbf{F}$ :

$$Y_0(\mathbf{F}) := Y_0 \left( \begin{bmatrix} \mathbf{F} \\ 0 \ 0 \end{bmatrix} \right). \quad (5.4.46)$$

Still, from (5.4.45) and  $Y_0(\mathbf{SI}) = Y_0(\mathbf{I})$  we have that  $Y_0(\mathbf{S}) = 0$  whenever  $\mathbf{S} \in O(2, 2) := \{\mathbf{X} \in \mathbb{M}^{2 \times 2} : \mathbf{X}^T \mathbf{X} = \mathbf{I}\}$ .

Let us take any  $2 \times 2$  matrix  $\mathbf{F}$  with both singular values less or equal to 1, and take its SVD decomposition

$$\mathbf{F} = \mathbf{U} \begin{bmatrix} \sigma_1(\mathbf{F}) & 0 \\ 0 & \sigma_2(\mathbf{F}) \end{bmatrix} \mathbf{V}^T. \quad (5.4.47)$$

Let us define functions  $\vartheta_\beta : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\beta = 1, 2$ , on  $[0, 1]$

$$\vartheta_\beta(t) = \begin{cases} (1 - \sigma_\beta(\mathbf{F}))t & \text{if } 0 \leq t \leq \frac{1+\sigma_\beta(\mathbf{F})}{2}, \\ (-1 - \sigma_\beta(\mathbf{F}))(t - 1) & \text{if } \frac{1+\sigma_\beta(\mathbf{F})}{2} \leq t \leq 1, \end{cases} \quad (5.4.48)$$

and extend it by periodicity to  $\mathbb{R}$ . Clearly, they belong to  $W^{1,\infty}([0, 1]^2; \mathbb{R}^2)$ . Let us define functions

$$\phi_0(\mathbf{V}^T \mathbf{y}) = \begin{bmatrix} \sigma_1(\mathbf{F})y_1 \\ \sigma_2(\mathbf{F})y_2 \end{bmatrix}, \quad \eta_n(\mathbf{V}^T \mathbf{y}) = \frac{1}{n} \begin{bmatrix} \vartheta_1(ny_1) \\ \vartheta_2(ny_2) \end{bmatrix}, \quad (5.4.49)$$

and  $\phi_n = \phi_0 + \eta_n$ , for  $n \geq 1$ . We have  $\nabla \phi_0 = \begin{bmatrix} \sigma_1(\mathbf{F}) & 0 \\ 0 & \sigma_2(\mathbf{F}) \end{bmatrix} \mathbf{V}^T$ . By direct calculations we see that the value of  $\nabla \phi_n(\mathbf{V}^T \mathbf{y}')$  for a.e.  $\mathbf{y}'$  is from the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{V}^T, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{V}^T, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{V}^T, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{V}^T \right\}, \quad (5.4.50)$$

which is a subset of  $O(2, 2)$ , so we have  $Y_0(\nabla \phi_n) = 0$ . Additionally,  $\phi_n \xrightarrow{*} \phi_0$  in  $W^{1,\infty}([0, 1]^2; \mathbb{R}^2)$ .

Since  $QY_0$  is quasiconvex, its integral is sequentially weak lower semicontinuous in  $W^{1,\infty}([0, 1]^2; \mathbb{R}^2)$  so we have

$$0 = \liminf_{n \rightarrow \infty} \int_{[0,1]^2} QY_0(\nabla \phi_n) d\mathbf{y}' \geq \int_{[0,1]^2} QY_0(\nabla \phi_0) d\mathbf{y}' = QY_0(\mathbf{F}) \geq 0, \quad (5.4.51)$$

thus  $QY_0(\mathbf{F}) = 0$ . The claim from the proposition is then consequence of Lemma 5.4.11 by which for any regular  $\mathbf{M}$  we have that  $\mathbf{F} \mapsto f(\mathbf{F})$  is quasiconvex iff  $\mathbf{F} \mapsto f(\mathbf{FM})$  is quasiconvex.  $\square$

**Lemma 5.4.11.** Let  $\mathbf{M} \in \mathbb{R}^{3 \times 3}$  be a regular matrix, and function  $F : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$ . Function  $\mathbf{X} \mapsto F(\mathbf{X})$  is quasiconvex if and only if  $\mathbf{X} \mapsto F(\mathbf{X}\mathbf{M})$  is quasiconvex.

*Proof.* Due to the regularity of  $\mathbf{M}$ , it sufficient to prove only one implication. Let  $F$  be quasiconvex, i.e. for each bounded  $D \subset \mathbb{R}^2$ , each  $\mathbf{X} \in \mathbb{R}^{3 \times 2}$  and each  $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^3)$  we have

$$F(\mathbf{X}) \leq \frac{1}{|D|} \int_D F(\mathbf{X} + \nabla' \varphi(\mathbf{x}')) d\mathbf{x}'. \quad (5.4.52)$$

Let us define  $G : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$ ,  $G(\mathbf{X}) = F(\mathbf{X}\mathbf{M})$ . We have to prove that  $G$  is quasiconvex, i.e. that the upper inequality holds for  $G$ , for a fixed set  $D$  and arbitrary  $\varphi$ . Let  $E := [0, 1]^2$ , and let  $\psi \in W^{1,\infty}(E, \mathbb{R}^3)$  be arbitrary. Let  $\mathbf{A} \in \mathbb{M}^{2 \times 2}$  be regular and  $\varphi \in W^{1,\infty}(\mathbf{A}(E), \mathbb{R}^3)$  be function such that  $\psi(\mathbf{z}') := \varphi(\mathbf{A}\mathbf{z}')$  for all  $\mathbf{z}' \in E$ . By quasiconvexity of  $F$  in point  $\mathbf{X}\mathbf{M}$  and change of variables we have

$$\begin{aligned} F(\mathbf{X}\mathbf{M}) &\leq \frac{1}{|\mathbf{A}(E)|} \int_{\mathbf{A}(E)} F(\mathbf{X}\mathbf{M} + \nabla' \varphi(\mathbf{x}')) d\mathbf{x}' \\ &= \left[ \begin{array}{c} \mathbf{x}' = \mathbf{A}\mathbf{z}' \\ d\mathbf{x}' = \mathbf{A}d\mathbf{z}' \end{array} \right] = \frac{1}{|\mathbf{A}(E)|} \int_E F(\mathbf{X}\mathbf{M} + \nabla' \varphi(\mathbf{A}\mathbf{z}')) |\det \mathbf{A}| d\mathbf{z}'. \end{aligned} \quad (5.4.53)$$

Due to definition of  $\varphi$ , we have  $\nabla' \psi(\mathbf{z}') = \nabla'(\mathbf{A}\mathbf{z}')\mathbf{A} \implies \nabla' \varphi(\mathbf{A}\mathbf{z}') = \nabla' \psi(\mathbf{z}')\mathbf{A}^{-1}$ .

Using this substitution and  $|\mathbf{A}(E)| = |\det \mathbf{A}| \cdot |E|$ , we have

$$\begin{aligned} F(\mathbf{X}\mathbf{M}) &\leq \frac{1}{|\det \mathbf{A}| \cdot |E|} \int_E F(\mathbf{X}\mathbf{M} + \nabla' \psi(\mathbf{z}')\mathbf{A}^{-1}) d\mathbf{z}' |\det \mathbf{A}| \\ &= \frac{1}{|E|} \int_E F(\mathbf{X}\mathbf{M} + \nabla' \psi(\mathbf{z}')\mathbf{A}^{-1}) d\mathbf{z}'. \end{aligned} \quad (5.4.54)$$

If we put  $\mathbf{A} := \mathbf{M}^{-1}$ , we finally get

$$G(\mathbf{X}) \leq \frac{1}{|E|} \int_E F(\mathbf{X} + \nabla' \psi(\mathbf{z}')) d\mathbf{z}'. \quad (5.4.55)$$

Since  $\psi$  was arbitrary,  $G$  is quasiconvex.  $\square$

In the sequel we determine the function  $QW_0$  for the St. Venant–Kirchhoff material.

**Lemma 5.4.12.** Let  $\hat{\mathbf{X}} = \begin{bmatrix} \mathbf{X} \\ \mathbf{x}^T \end{bmatrix} \in \mathbb{R}^{3 \times 2}$ , and let  $\mathbf{X}_{sym}, \mathbf{X}_{skew}$  be the symmetric and skew-symmetric parts of the matrix  $\mathbf{X}$ . Then it holds

$$\begin{aligned} \mathcal{C}_m^2 \left( \begin{bmatrix} \mathbf{A}^c \mathbf{X} \\ \mathbf{x}^T \end{bmatrix} \right) &= \frac{2\lambda\mu}{\lambda + 2\mu} (\text{tr}(\sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c}))^2 + 2\mu \left\| \sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c} \right\|_F^2 \\ &\quad + 2\mu \left\| \sqrt{\mathbf{A}^c} \mathbf{X}_{skew} \sqrt{\mathbf{A}^c} \right\|_F^2 + \mu \mathbf{A}^c \mathbf{x} \cdot \mathbf{x}. \end{aligned}$$

*Proof.* From the definition of the function  $\mathcal{C}_m^2$  from Remark 5.2.2 we have  $\mathcal{C}_m^2(\hat{\mathbf{Y}}) = \mathcal{C}_m(\hat{\mathbf{Y}}) \cdot \hat{\mathbf{Y}}$ , where

$$\mathcal{C}_m(\hat{\mathbf{Y}}) \cdot \hat{\mathbf{Y}} = \frac{2\lambda\mu}{\lambda + 2\mu} \text{tr}(\mathbf{Y})^2 + 2\mu \text{tr}(\mathbf{A}^c \mathbf{Y}^T \mathbf{A}^c \mathbf{Y}) + \mu \mathbf{A}^c \mathbf{y} \cdot \mathbf{y},$$

for each  $\hat{\mathbf{Y}} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{y}^T \end{bmatrix} \in \mathbb{R}^{3 \times 2}$ . For  $\mathbf{Y} = \mathbf{A}^c \mathbf{X}$  we calculate

$$\begin{aligned} \text{tr}(\mathbf{A}^c \mathbf{X}) &= \text{tr}(\sqrt{\mathbf{A}^c} \mathbf{X} \sqrt{\mathbf{A}^c}) \\ &= \text{tr}(\sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c}) + \text{tr}(\sqrt{\mathbf{A}^c} \mathbf{X}_{skew} \sqrt{\mathbf{A}^c}) = \text{tr}(\sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c}), \end{aligned}$$

where we have used the fact that if  $\mathbf{Y}$  is (skew-)symmetric, then such is  $\sqrt{\mathbf{A}^c} \mathbf{Y} \sqrt{\mathbf{A}^c}$  as well, and that the trace of skew-symmetric matrices are zero. Secondly, we have

$$\text{tr}(\mathbf{A}^c \mathbf{X}^T \mathbf{A}^c \mathbf{A}^c \mathbf{A}^c \mathbf{X}) = \text{tr}((\sqrt{\mathbf{A}^c} \mathbf{X} \sqrt{\mathbf{A}^c})^T (\sqrt{\mathbf{A}^c} \mathbf{X} \sqrt{\mathbf{A}^c})) = \|\sqrt{\mathbf{A}^c} \mathbf{X} \sqrt{\mathbf{A}^c}\|_F^2.$$

Here we again use that matrices  $\sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c}$  and  $\sqrt{\mathbf{A}^c} \mathbf{X}_{skew} \sqrt{\mathbf{A}^c}$  are symmetric and skew-symmetric respectively, and consequently orthogonal, so we have

$$\text{tr}(\mathbf{A}^c \mathbf{X}^T \mathbf{A}^c \mathbf{A}^c \mathbf{A}^c \mathbf{X}) = \|\sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c}\|_F^2 + \|\sqrt{\mathbf{A}^c} \mathbf{X}_{skew} \sqrt{\mathbf{A}^c}\|_F^2.$$

This is enough to conclude the claim of the lemma.  $\square$

**Lemma 5.4.13.** It holds

$$\begin{aligned} W_0(\mathbf{F}) &= \frac{\lambda\mu}{4(\lambda + 2\mu)} (\text{tr}(\sqrt{\mathbf{A}^c} (\mathbf{F}^T \mathbf{F} - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}) \sqrt{\mathbf{A}^c}))^2 \\ &\quad + \frac{\mu}{4} \left\| \sqrt{\mathbf{A}^c} (\mathbf{F}^T \mathbf{F} - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}) \sqrt{\mathbf{A}^c} \right\|_F^2. \end{aligned} \quad (5.4.56)$$

*Proof.* In the view of Lemma 5.3.2 ( $\mathbf{F} = \nabla' \boldsymbol{\psi}$ ), we use Lemma 5.4.12 with matrices

$$\mathbf{X}_{sym} = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}), \quad \mathbf{X}_{skew} = \frac{1}{2} (\mathbf{F}^T \mathbf{S} \nabla' \boldsymbol{\theta} - \nabla' \boldsymbol{\theta}^T \mathbf{S}^T \mathbf{F}), \quad \mathbf{x} = \mathbf{F}^T \mathbf{S} \mathbf{a}_3$$

to obtain

$$\begin{aligned} w_m^{STVK}(\mathbf{S}, \mathbf{F}) &= \frac{1}{2} \mathcal{C}_m^2 \left( \begin{bmatrix} \mathbf{A}^c (\mathbf{X}_{sym} + \mathbf{X}_{skew}) \\ \mathbf{x}^T \end{bmatrix} \right) \\ &= \frac{\lambda\mu}{\lambda + 2\mu} (\text{tr}(\sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c}))^2 + \mu \left\| \sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c} \right\|_F^2 \\ &\quad + \mu \left\| \sqrt{\mathbf{A}^c} \mathbf{X}_{skew} \sqrt{\mathbf{A}^c} \right\|_F^2 + \frac{\mu}{2} \mathbf{A}^c \mathbf{x} \cdot \mathbf{x}. \end{aligned}$$

To calculate  $W_0(\mathbf{F})$  we have to minimize  $w_m^{STVK}$  in terms of  $\mathbf{S}$ , which appears only in  $\mathbf{X}_{skew}$  and  $\mathbf{x}$ . Those terms

$$\mu \left\| \sqrt{\mathbf{A}^c} \mathbf{X}_{skew} \sqrt{\mathbf{A}^c} \right\|_F^2 \text{ and } \frac{\mu}{2} \mathbf{A}^c \mathbf{x} \cdot \mathbf{x} \quad (5.4.57)$$

are nonnegative, so they are bounded from below by zero. However, by Lemma 5.5.4 for any  $\mathbf{F}$  there exists a choice  $\mathbf{S}$  such that these terms are zero. Thus we have

$$\begin{aligned} W_0(\mathbf{F}) &= \inf_{\mathbf{S} \in \text{SO}(3)} w_m^{STVK}(\mathbf{S}, \mathbf{F}) \\ &= \frac{\lambda\mu}{\lambda + 2\mu} (\text{tr}(\sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c}))^2 + \mu \left\| \sqrt{\mathbf{A}^c} \mathbf{X}_{sym} \sqrt{\mathbf{A}^c} \right\|_F^2, \end{aligned}$$

which gives the statement after inserting  $\mathbf{X}_{sym}$ .  $\square$

Since we have that  $\nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta} = \mathbf{A}_c$ , we have  $\sqrt{\mathbf{A}^c} \mathbf{A}_c \sqrt{\mathbf{A}^c} = \mathbf{I}$ , and

$$\sqrt{\mathbf{A}^c} (\mathbf{F}^T \mathbf{F} - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}) \sqrt{\mathbf{A}^c} = (\mathbf{F} \sqrt{\mathbf{A}^c})^T (\mathbf{F} \sqrt{\mathbf{A}^c}) - \mathbf{I}. \quad (5.4.58)$$

So it is natural to define function  $Y_0$  as  $Y_0(\mathbf{F} \sqrt{\mathbf{A}^c}) = W_0(\mathbf{F})$ . As we already used, seeking the quasiconvex envelope of function  $Y_0$  is equivalent to seeking quasiconvex envelope of function  $W_0$  by Lemma 5.4.11.

**Proposition 5.4.14.** For  $w_m = w_m^{STVK}$ , we have

$$QW_0(\mathbf{F}) = \begin{cases} 0 & \sigma_2(\mathbf{F} \sqrt{\mathbf{A}^c}) \leq \sigma_1(\mathbf{F} \sqrt{\mathbf{A}^c}) \leq 1, \\ \frac{E}{4} (\sigma_1(\mathbf{F} \sqrt{\mathbf{A}^c}) - 1)^2 & \sigma_1(\mathbf{F} \sqrt{\mathbf{A}^c}) \geq 1 \text{ and} \\ & \sigma_2(\mathbf{F} \sqrt{\mathbf{A}^c}) + \nu \sigma_1(\mathbf{F} \sqrt{\mathbf{A}^c}) \leq 1 + \nu, \\ W_0(\mathbf{F}) & \text{otherwise.} \end{cases} \quad (5.4.59)$$

*Proof.* As said before the lemma, we observe  $Y_0$  such that  $Y_0(\mathbf{F} \sqrt{\mathbf{A}^c}) = W_0(\mathbf{F})$ , or in other words

$$Y_0(\mathbf{F}) = \frac{\lambda\mu}{4(\lambda + 2\mu)} (\text{tr}(\mathbf{F}^T \mathbf{F} - \mathbf{I}))^2 + \frac{\mu}{4} \left\| \mathbf{F}^T \mathbf{F} - \mathbf{I} \right\|_F^2. \quad (5.4.60)$$

By denoting by  $W_0^{LD-R}$  and  $QW_0^{LD-R}$  functions from Proposition 16 from [51], we have

$$QW_0^{LD-R}(\mathbf{F}) \leq Y_0(\mathbf{F}) \leq W_0^{LD-R}(\mathbf{F}). \quad (5.4.61)$$

from where we have

$$QY_0(\mathbf{F}) = QW_0^{LD-R}(\mathbf{F}) = \begin{cases} 0 & \sigma_2(\mathbf{F}) \leq \sigma_1(\mathbf{F}) \leq 1, \\ \frac{E}{4}(\sigma_1(\mathbf{F}) - 1)^2 & \sigma_1(\mathbf{F}) \geq 1 \text{ and} \\ & \sigma_2(\mathbf{F}) + \nu\sigma_1(\mathbf{F}) \leq 1 + \nu, \\ Y_0(\mathbf{F}) & \text{otherwise.} \end{cases} \quad (5.4.62)$$

Then due to  $Y_0(\mathbf{F}\sqrt{\mathbf{A}^c}) = W_0(\mathbf{F})$  we have  $QY_0(\mathbf{F}\sqrt{\mathbf{A}^c}) = QW_0(\mathbf{F})$ , which completes the proof.  $\square$

**Remark 5.4.15.** Thus we have  $QW_0(\mathbf{F}) = QY_0(\mathbf{F}\sqrt{\mathbf{A}^c})$  and further  $QY_0$  actually depends on singular values of  $\mathbf{F}\sqrt{\mathbf{A}^c}$ . In the proof of Lemma 5.4.9 we actually proved that for matrices  $\mathbf{X}$  and  $\mathbf{Y}$  such that  $\mathbf{X} = \mathbf{Y} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$  the two largest singular values of  $\mathbf{Y}$  are singular values of  $\mathbf{X}\sqrt{\mathbf{A}_c}$ . Thus, we obtain that in the case of St. Venant–Kirchhoff material obtained limit model in the membrane case coincides with the membrane shell model from [51].

#### 5.4.4. Flexural shell model - convergence for the case $\alpha = 2$

In this subsection we prove the claims of Theorem 5.4.1.c) and Theorem 5.4.2.d). The goal is to obtain a limit and compare the result with the flexural shell model obtained as the  $\Gamma$ -limit of the three-dimensional elastic energy in [40]. It is a generalization of result for the planar middle surface geometry presented in Theorem 3.2.1.d).

In order to prove this result we use a result from [40] in which the authors state that if the measure of distance of deformation gradient to set  $\text{SO}(3)$  tends to zero sufficiently fast (with respect to the thickness  $h$ ), then the deformation gradient tends to a function with values in  $\text{SO}(3)$ . Since our functions are defined on  $\omega$ , domain in  $\mathbb{R}^2$ , we associate with them functions on the 3d domain  $\Omega = \omega \times [-1/2, 1/2]$ . For a family  $(\psi_h, \mathbf{S}_h)_{h>0}$  we define functions

$$\begin{aligned} \Theta_h : \Omega &\rightarrow \mathbb{R}^3, & \Theta_h(x_1, x_2, x_3) &= \boldsymbol{\theta}(x_1, x_2) + hx_3\mathbf{a}_3, \\ \Psi_h : \Omega &\rightarrow \mathbb{R}^3, & \Psi_h(x_1, x_2, x_3) &= \boldsymbol{\psi}_h(x_1, x_2) + hx_3\mathbf{S}_h(x_1, x_2)\mathbf{a}_3, \\ u_h : \Theta_h(\Omega) &\rightarrow \mathbb{R}^3, & u_h &= \Psi_h \circ \Theta_h^{-1}. \end{aligned} \quad (5.4.63)$$

Note that

$$\begin{aligned}
 \nabla u_h \circ \Theta_h &= \begin{bmatrix} \nabla' \psi_h + hx_3 \nabla'(\mathbf{S}_h \mathbf{a}_3) & h \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \nabla' \theta + hx_3 \nabla' \mathbf{a}_3 & h \mathbf{a}_3 \end{bmatrix}^{-1} \\
 &= \begin{bmatrix} \nabla' \psi_h + hx_3 \nabla'(\mathbf{S}_h \mathbf{a}_3) & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \mathbf{D} \left( \begin{bmatrix} \nabla' \theta + hx_3 \nabla' \mathbf{a}_3 & \mathbf{a}_3 \end{bmatrix} \mathbf{D} \right)^{-1} \\
 &= \begin{bmatrix} \nabla' \psi_h + hx_3 \nabla'(\mathbf{S}_h \mathbf{a}_3) & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \nabla' \theta + hx_3 \nabla' \mathbf{a}_3 & \mathbf{a}_3 \end{bmatrix}^{-1},
 \end{aligned} \tag{5.4.64}$$

where  $\mathbf{D} = \text{diag}(1, 1, h)$ . From our assumptions:  $\Theta \in C^2(\omega; \mathbb{R}^3)$ ,  $\Psi_h \in W^{1,2}(\omega; \mathbb{R}^3)$ ,  $u_h \in W^{1,2}(\omega; \mathbb{R}^3)$ .

Directly from Lemma 5.4.3 we have the following result.

**Corollary 5.4.16.** Let  $\alpha > 0$ . Then there are constants  $c > 0$  and  $C$ , such that for all  $\psi, \mathbf{S} \in \mathcal{A}^N$  for all  $h > 0$  small enough we have the estimate

$$\begin{aligned}
 \|\psi\|_{L^4(\omega; \mathbb{R}^3)}^4 + \|\mathbf{S}\|_{L^2(\omega; \text{SO}(3))}^2 + J_h^{2d, \alpha}(\psi, \mathbf{S}) &\geq \\
 \frac{c}{h^\alpha} \Big( \|\nabla' \psi^T \nabla' \psi - \nabla' \theta^T \nabla' \theta\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 & \\
 + \|\nabla' \psi^T \mathbf{S} \nabla' \theta - \nabla' \theta^T \mathbf{S}^T \nabla' \psi\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + \|\mathbf{S} \mathbf{a}_3 \cdot \partial_\beta \psi\|_{L^2(\omega; \mathbb{R}^{3 \times 2})}^2 & \\
 + c \left( \|\psi\|_{W^{1,4}(\omega; \mathbb{R}^3)}^4 + \|\mathbf{S}\|_{L^2(\omega; \text{SO}(3))}^2 + \chi_{\{\alpha \geq 2\}} \|\nabla' \mathbf{S}\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}^2 \right) &- C.
 \end{aligned} \tag{5.4.65}$$

Next we prove the lower semicontinuity property of the flexural energy.

**Lemma 5.4.17.** Let  $\mathbf{S}_n \rightharpoonup \mathbf{S}_0$  in  $W^{1,2}(\omega; \mathbb{R}^{3 \times 3})$ , with  $\mathbf{S}_n, \mathbf{S}_0 \in W^{1,2}(\omega; \mathbb{R}^{3 \times 3})$ , for all  $n \geq 1$ . Then

$$\liminf_{n \rightarrow \infty} \int_{\omega} q_f(\mathbf{S}_n, \nabla' \mathbf{S}_n) \sqrt{a} d\mathbf{x}' \geq \int_{\omega} q_f(\mathbf{S}_0, \nabla' \mathbf{S}_0) \sqrt{a} d\mathbf{x}'. \tag{5.4.66}$$

*Proof.* Let us define  $F : \omega \times \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 6} \rightarrow \mathbb{R}$ ,  $F(\mathbf{x}', \mathbf{R}, \Sigma) = q_f(\mathbf{x}')(\mathbf{R}, \Sigma) \sqrt{a(\mathbf{x}')}.$

From the growth condition given in assumption N2) we have

$$0 \leq F(\mathbf{x}', \mathbf{R}, \Sigma) \leq C(1 + \|\mathbf{R}^T \Sigma\|_F^2) \leq C(1 + \|\mathbf{R}\|_F^2)(1 + \|\Sigma\|_F^2). \tag{5.4.67}$$

Since  $Q_f$  is convex function, function  $F$  is trivially quasiconvex in the last variable. According to [32], Theorem 8.11.,

$$\mathbf{S} \mapsto \int_{\omega} q_f(\mathbf{S}, \nabla' \mathbf{S}) \sqrt{a} d\mathbf{x}' \tag{5.4.68}$$

is weakly lower semicontinuous in  $W^{1,2}(\omega; \mathbb{R}^{3 \times 3})$ , which completes the proof.  $\square$



**Lemma 5.4.18.** Let  $\alpha \in \langle 0, 2 \rangle$ , and let  $(\psi_h, \mathbf{S}_h)_{h>0} \subset \mathcal{A}^N$  satisfy bounds

$$\|\nabla' \psi_h\|_{W^{1,4}(\omega; \mathbb{R}^3)} \leq M, \quad h \|\nabla' \mathbf{S}_h\|_{L^2(\omega; \mathbb{R}^{3 \times 6})} \leq M h^{\alpha/2} \quad (5.4.69)$$

for a real  $M > 0$ . Let

$$\mathcal{I}_1 := \|\nabla' \psi_h^T \nabla' \psi_h - \nabla' \theta^T \nabla' \theta\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + \|\mathbf{S}_h \mathbf{a}_3 \cdot \partial_1 \psi_h\|_{L^2(\omega; \mathbb{R})}^2 + \|\mathbf{S}_h \mathbf{a}_3 \cdot \partial_2 \psi_h\|_{L^2(\omega; \mathbb{R})}^2, \quad (5.4.70)$$

and

$$\mathcal{I}_2 := \frac{1}{h} \int_{\Theta_h(\Omega)} \text{dist}^2(\nabla u_h, \text{SO}(3)) d\mathbf{x}, \quad (5.4.71)$$

where  $u_h$  is defined in (5.4.63). Then there exist positive constants  $c, C$  (depending only on  $M, \alpha$  and properties of  $\theta$ ) such that

$$c\mathcal{I}_2 - \mathcal{O}(h^\alpha) \leq \mathcal{I}_1 \leq C\mathcal{I}_2 + \mathcal{O}(h^\alpha). \quad (5.4.72)$$

*Proof.* Let us start the proof with a claim we will often use in several following proofs. Since rotations  $\text{SO}(3)$  is a bounded set in  $\mathbb{M}^{3 \times 3}$  and  $\mathbf{S}_h$  are measurable functions with  $\mathbf{S}_h(\mathbf{x}') \in \text{SO}(3)$  a.e.,  $(\mathbf{S}_h)_{h>0}$  are uniformly bounded in  $L^\infty(\omega; \mathbb{R}^{3 \times 3})$ , and consequently there is a constant  $M' > 0$  such that

$$\|\mathbf{S}_h\|_{L^2(\omega; \text{SO}(3))} \leq M' \quad (5.4.73)$$

for all  $h > 0$  (actually the constant  $M'$  depends only on the domain  $\omega$ ). Also, note that (5.4.69) implies uniform boundedness of  $(\nabla' \psi_h)_{h>0}$  in  $L^2(\omega; \mathbb{R}^{3 \times 2})$ .

We apply Lemma 5.5.7 for matrices  $\mathbf{P}_h = \begin{bmatrix} \nabla' \psi_h & \mathbf{a}_3^{\psi_h} \end{bmatrix}$ ,  $\mathbf{M}_h = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ ,  $\mathbf{Q}_h = \mathbf{S}_h$ ,  $\mathbf{T}_h = \begin{bmatrix} \partial_1 \psi_h & \partial_2 \psi_h & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix}$  and  $\mathbf{N}_h = \mathbf{S}_h \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ . Bounds in the lemma for our matrices depend on uniform bounds for matrices

$$\mathbf{T}_h = \begin{bmatrix} \partial_1 \psi_h & \partial_2 \psi_h & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix}, \quad \mathbf{N}_h = \mathbf{S}_h \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_h^{-1} = \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix}^T \mathbf{S}_h^T. \quad (5.4.74)$$

Matrices  $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$  and  $\begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix}^T$  depend only on the geometry of undeformed shell. Since we assumed  $\theta \in C^2(\omega; \mathbb{R}^3)$ , those matrices are bounded in  $L^\infty(\omega; \mathbb{R}^{3 \times 3})$ . So, by using (5.4.73) we obtain uniform bounds

$$\|\mathbf{N}_h\|_{L^2(\omega; \mathbb{R}^{3 \times 3})}, \|\mathbf{N}_h^{-1}\|_{L^2(\omega; \mathbb{R}^{3 \times 3})} \leq M''.$$

On the other hand, matrices  $\mathbf{T}_h = \begin{bmatrix} \partial_1 \psi_h & \partial_2 \psi_h & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix}$  have positive determinant (due to  $(\psi_h, \mathbf{S}_h) \in \mathcal{A}^N$ ) and are uniformly bounded in  $L^2(\omega; \mathbb{R}^{3 \times 3})$  (due to (5.4.69) and (5.4.73)). Thus we are indeed able to apply Lemma 5.5.7, integrated over  $\omega$ , and we obtain that both bounds with constants independent on  $(\psi_h, \mathbf{S}_h)_{h>0}$ . Thus we conclude that  $\mathcal{I}_1$  can be bounded from below and above by constants multiplying

$$\begin{aligned} \mathcal{I}_3 &:= \int_{\omega} \text{dist}^2 \left( \begin{bmatrix} \nabla' \psi_h & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \mathbf{S}_h \mathbf{a}_1 & \mathbf{S}_h \mathbf{a}_2 & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix}^{-1}, \text{SO}(3) \right) \sqrt{a} d\mathbf{x}' \\ &= \int_{\Omega} \text{dist}^2 \left( \begin{bmatrix} \nabla' \psi_h & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}^{-1}, \text{SO}(3) \right) \sqrt{a} d\mathbf{x}. \end{aligned} \quad (5.4.75)$$

We will now apply Lemma 5.5.2 for matrices

$$\begin{aligned} &\begin{bmatrix} \nabla' \psi_h & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}^{-1} \\ &\text{and } \begin{bmatrix} \nabla' \psi_h + hx_3 \nabla'(\mathbf{S}_h \mathbf{a}_3) & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \nabla' \boldsymbol{\theta} + hx_3 \nabla' \mathbf{a}_3 & \mathbf{a}_3 \end{bmatrix}^{-1}. \end{aligned} \quad (5.4.76)$$

Since matrices  $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}^{-1}$  and  $\begin{bmatrix} x_3 \nabla' \mathbf{a}_3 & 0 \end{bmatrix}^{-1}$  are only undeformed geometry dependant, by similar arguments as before they are bounded in  $L^\infty(\omega; \mathbb{R}^{3 \times 3})$  by a constant  $C_{\text{geo}}$ . We proved that matrices  $\begin{bmatrix} \partial_1 \psi_h & \partial_2 \psi_h & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix}$  (denoted by  $\mathbf{T}_h$  in application of the previous lemma) are uniformly bounded in  $L^2(\omega; \mathbb{R}^{3 \times 3})$ . We finally apply Lemma 5.5.2 for matrices in (5.4.76) and obtain that  $L^2(\omega; \mathbb{R}^{3 \times 3})$  norm of their difference is bounded by

$$C \left( \left\| \begin{bmatrix} \nabla' \psi_h & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \right\|_{L^2(\omega; \mathbb{R}^{3 \times 3})}^2 C_{\text{geo}}^6 h^2 + \left\| \begin{bmatrix} hx_3 \nabla'(\mathbf{S}_h \mathbf{a}_3) & 0 \end{bmatrix} \right\|_{L^2(\omega; \mathbb{R}^{3 \times 3})}^2 C_{\text{geo}}^2 \right). \quad (5.4.77)$$

Also, we note that  $\partial_\beta(\mathbf{S}_h \mathbf{a}_3) = \partial_\beta \mathbf{S}_h \mathbf{a}_3 + \mathbf{S}_h \partial_\beta \mathbf{a}_3$ . Derivatives of  $\mathbf{a}_3$  are also bounded by  $C_{\text{geo}}$ , while for derivatives of  $\mathbf{S}_h$  we use bound (5.4.69) to conclude that the expression in (5.4.77) is bounded from above by  $\mathcal{O}(h^\alpha)$ . After the change of variables we further have that  $\mathcal{I}_3$  and

$$\begin{aligned} \mathcal{I}_4 &:= \\ &\int_{\Omega} \text{dist}^2 \left( \begin{bmatrix} \nabla' \psi_h + hx_3 \nabla'(\mathbf{S}_h \mathbf{a}_3) & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \nabla' \boldsymbol{\theta} + hx_3 \nabla' \mathbf{a}_3 & \mathbf{a}_3 \end{bmatrix}^{-1}, \text{SO}(3) \right) \sqrt{a} d\mathbf{x} \\ &= \int_{\boldsymbol{\Theta}_h(\Omega)} \det \nabla \boldsymbol{\Theta}_h^{-1} \text{dist}^2(\nabla u_h, \text{SO}(3)) dz \end{aligned} \quad (5.4.78)$$

also differ by  $\mathcal{O}(h^\alpha)$ . Finally, since

$$\det \nabla \Theta_h^{-1} = \det \begin{bmatrix} \nabla' \boldsymbol{\theta} + hx_3 \nabla' \mathbf{a}_3 & h \mathbf{a}_3 \end{bmatrix}^{-1} = \frac{1}{h \det \begin{bmatrix} \nabla' \boldsymbol{\theta} + hx_3 \nabla' \mathbf{a}_3 & \mathbf{a}_3 \end{bmatrix}}$$

can be bounded from above and below by a constant multiplying  $h^{-1}$ , we obtain the claim by comparing  $\mathcal{I}_1, \mathcal{I}_3, \mathcal{I}_4$  and  $\mathcal{I}_2$ .  $\square$

*Proof of Theorem 5.4.1.c).* We prove the theorem by the definition of  $\Gamma$ -limit. It is clear that the "lim sup" part is obtained for the constant sequence  $(\boldsymbol{\psi}, \mathbf{S})$  in the set of admissible functions.

Let us prove the "lim inf" part: let us take any  $(\boldsymbol{\psi}_h, \mathbf{S}_h) \rightarrow (\boldsymbol{\psi}_0, \mathbf{S}_0)$  in  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  with  $\liminf J_h^{2d,2}(\boldsymbol{\psi}_h, \mathbf{S}_h) < +\infty$ . From coercivity estimate from Corollary 5.4.16 we have

$$\|\mathbf{S}_h\|_{W^{1,2}(\omega; \text{SO}(3))}, \quad \|\boldsymbol{\psi}_h\|_{W^{1,4}(\omega; \mathbb{R}^3)} \leq C \quad (5.4.79)$$

and

$$\begin{aligned} & \|\nabla' \boldsymbol{\psi}_h^T \nabla' \boldsymbol{\psi}_h - \nabla' \boldsymbol{\theta}^T \nabla' \boldsymbol{\theta}\|_{L^2(\omega; \mathbb{R}^3)}^2 + \|\nabla' \boldsymbol{\psi}_h^T \mathbf{S}_h \nabla' \boldsymbol{\theta} - \nabla' \boldsymbol{\theta}^T \mathbf{S}_h^T \nabla' \boldsymbol{\psi}_h\|_{L^2(\omega; \mathbb{R}^3)}^2 \\ & + \|\mathbf{S}_h \mathbf{a}_3 \cdot \partial_1 \boldsymbol{\psi}_h\|_{L^2(\omega; \mathbb{R}^3)}^2 + \|\mathbf{S}_h \mathbf{a}_3 \cdot \partial_2 \boldsymbol{\psi}_h\|_{L^2(\omega; \mathbb{R}^3)}^2 \leq Ch^2. \end{aligned} \quad (5.4.80)$$

Thus there is a subsequence of  $(\boldsymbol{\psi}_h, \mathbf{S}_h)$  that converges weakly to  $(\boldsymbol{\psi}_0, \mathbf{S}_0)$  in  $W^{1,4}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3))$ . Due to the uniqueness of limits (weak  $W^{1,4}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3))$  and strong  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  limit), the whole sequence  $(\boldsymbol{\psi}_h, \mathbf{S}_h)$  converges weakly to  $(\boldsymbol{\psi}_0, \mathbf{S}_0)$  in  $W^{1,4}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3))$ .

If we prove that  $(\boldsymbol{\psi}_0, \mathbf{S}_0) \in \mathcal{A}^{\pm F}$  ( $\mathcal{A}^F$ , if we have Dirichlet boundary conditions prescribed on a set of positive capacity), we will be done. Namely, since  $w_m$  is nonnegative and  $q_f$  is sequentially weakly lower semicontinuous by Lemma 5.4.17, we will indeed have  $\liminf_{h \rightarrow 0} J_h^{2d,2}(\boldsymbol{\psi}_h, \mathbf{S}_h) \geq J_0^{2d,2}(\boldsymbol{\psi}_0, \mathbf{S}_0)$ .

From the last three terms in (5.4.80) and Lemma 5.4.5 we conclude that

$$\partial_1 \boldsymbol{\psi}_0 \cdot \mathbf{S}_0 \mathbf{a}_2 - \mathbf{S}_0 \mathbf{a}_1 \cdot \partial_2 \boldsymbol{\psi}_0 = 0, \quad \mathbf{S}_0 \mathbf{a}_3 \cdot \partial_\beta \boldsymbol{\psi}_0 = 0, \quad \beta \in \{1, 2\}. \quad (5.4.81)$$

By Lemma 5.4.18 (for  $\alpha = 2$ , from (5.4.79) and (5.4.80)) we obtain

$$\int_{\Theta_h(\Omega)} \text{dist}^2(\nabla u_h, \text{SO}(3)) d\mathbf{x} \leq Ch^3. \quad (5.4.82)$$

Now we are in the assumptions of Theorem 1 from [40] and conclude that there is  $\mathbf{R} \in W^{1,2}(\Omega, \text{SO}(3))$  such that  $\partial_3 \mathbf{R} = 0$  and

$$\nabla u_h \circ \Theta_h = \begin{bmatrix} \nabla' \psi_h + hx_3 \nabla'(\mathbf{S}_h \mathbf{a}_3) & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \nabla' \theta + hx_3 \nabla' \mathbf{a}_3 & \mathbf{a}_3 \end{bmatrix}^{-1} \rightarrow \mathbf{R} \quad (5.4.83)$$

in  $L^2(\Omega, \mathbb{R}^3)$ . By same reasoning as in the proof of Lemma 5.4.18, from Lemma 5.5.2 we have that the difference

$$\begin{aligned} & \begin{bmatrix} \nabla' \psi_h & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}^{-1} \\ & - \begin{bmatrix} \nabla' \psi_h + hx_3 \nabla'(\mathbf{S}_h \mathbf{a}_3) & \mathbf{S}_h \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \nabla' \theta + hx_3 \nabla' \mathbf{a}_3 & \mathbf{a}_3 \end{bmatrix}^{-1} \end{aligned} \quad (5.4.84)$$

tends to 0 in  $L^2(\omega; \mathbb{R}^{3 \times 3})$  as  $h \rightarrow 0$ . The weak convergence  $(\psi_h, \mathbf{S}_h) \rightharpoonup (\psi_0, \mathbf{S}_0)$  in  $W^{1,4}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3))$  implies that the first term in the difference from above converges to  $\begin{bmatrix} \nabla' \psi_0 & \mathbf{S}_0 \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}^{-1}$ . By (5.4.83) the second term tends to  $\mathbf{R}$ . Therefore, by the uniqueness of the limits we conclude that

$$\mathbf{R} = \begin{bmatrix} \nabla' \psi_0 & \mathbf{S}_0 \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}^{-1} \in W^{1,2}(\Omega, \text{SO}(3)). \quad (5.4.85)$$

Since any two functions  $\mathbf{Q}_1, \mathbf{Q}_2$  from  $W^{1,2}(\Omega, \text{SO}(3))$  specially belong to  $L^\infty(\Omega; \mathbb{R}^{3 \times 3})$  (due to boundedness of  $\text{SO}(3)$  in  $\mathbb{M}^{3 \times 3}$ ), we have

$$\begin{aligned} & \|\partial_\beta(\mathbf{Q}_1 \mathbf{Q}_2)\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} \\ & \leq C \left( \|\partial_\beta \mathbf{Q}_1\|_{L^\infty(\Omega; \mathbb{R}^{3 \times 3})} \|\mathbf{Q}_2\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} + \|\mathbf{Q}_1\|_{L^\infty(\Omega; \mathbb{R}^{3 \times 3})} \|\partial_\beta \mathbf{Q}_2\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} \right), \end{aligned}$$

so their product is also in  $W^{1,2}(\Omega, \text{SO}(3))$ . Thus we conclude

$$\mathbf{R} \mathbf{S}_0^T = \begin{bmatrix} \nabla' \psi_0 & \mathbf{S}_0 \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \mathbf{S}_0 \mathbf{a}_1 & \mathbf{S}_0 \mathbf{a}_2 & \mathbf{S}_0 \mathbf{a}_3 \end{bmatrix}^{-1} \in W^{1,2}(\Omega, \text{SO}(3)), \quad (5.4.86)$$

Since  $\mathbf{R} \mathbf{S}_0^T \in \text{SO}(3)$  a.e., we conclude that both matrices in the product from above are regular a.e. Specially,  $\nabla' \psi_0$  has a full rank and  $\mathbf{a}_3^{\psi_0}$  exists a.e. By Lemma 5.5.3 for  $\mathbf{P} = \begin{bmatrix} \nabla' \psi_0 & \mathbf{a}_3^{\psi_0} \end{bmatrix}$ ,  $\mathbf{Q} = \mathbf{S}_0$ ,  $\mathbf{M} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ , from by (5.4.81) we obtain that  $\mathbf{S}_0 \mathbf{a}_3 = \mathbf{a}_3^{\psi_0}$ . Again by using (5.4.81) we can apply Lemma 5.5.6 for matrices  $\mathbf{Q} = \mathbf{S}_0$ ,  $\mathbf{M} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ ,  $\mathbf{P} = \begin{bmatrix} \nabla' \psi_0 & \mathbf{a}_3^{\psi_0} \end{bmatrix}$  and  $\mathbf{T} = \begin{bmatrix} \nabla' \psi_0 & \mathbf{S}_0 \mathbf{a}_3 \end{bmatrix}$  to obtain that either  $\mathbf{R} \mathbf{S}_0^T = \mathbf{I}$  or  $\mathbf{R} \mathbf{S}_0^T = \text{diag}(-1, -1, 1)$ , pointwise. Since  $\mathbf{R} \mathbf{S}_0^T \in W^{1,2}(\Omega, \text{SO}(3))$  there are only two possibilities: the plus sign over the whole domain  $\omega$  or the minus sign over the whole domain  $\omega$ , thus  $(\psi, \mathbf{S}) \in \mathcal{A}^{\pm F}$ .

In the case of prescribed Dirichlet boundary condition, since on the boundary we have the plus sign this implies  $\nabla' \boldsymbol{\psi} = \mathbf{S} \nabla' \boldsymbol{\theta}$ , so we have  $(\boldsymbol{\psi}_0, \mathbf{S}_0) \in \mathcal{A}^F$ , and the proof is complete.  $\square$

**Proposition 5.4.19.** Let us assume  $\gamma_0 = \emptyset$ , i.e. there is no Dirichlet condition.

- a) Let  $\mathbf{S} \in W^{1,2}(\omega; \text{SO}(3))$ . Then there exists  $\boldsymbol{\psi} \in W^{2,2}(\omega; \mathbb{R}^3)$  such that  $(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^{\pm F}$  if and only if  $\mathbf{S}$  satisfies

$$\partial_1 \mathbf{S} \mathbf{a}_2 = \partial_2 \mathbf{S} \mathbf{a}_1. \quad (5.4.87)$$

In that case all such functions  $\boldsymbol{\psi}$  are of the form  $\boldsymbol{\psi}_1 + \mathbf{c}$  or  $\boldsymbol{\psi}_2 + \mathbf{c}$ , where  $\boldsymbol{\psi}_1 = -\boldsymbol{\psi}_2$ , and  $\mathbf{c} \in \mathbb{R}^3$  is an arbitrary constant vector.

- b) Let  $\boldsymbol{\psi} \in W^{2,2}(\omega; \mathbb{R}^3)$ . Then there exists  $\mathbf{S} \in W^{1,2}(\omega; \text{SO}(3))$  such that  $(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^{\pm F}$  if and only if  $\boldsymbol{\psi}$  satisfies

$$\nabla \boldsymbol{\psi}^T \nabla \boldsymbol{\psi} = \nabla \boldsymbol{\theta}^T \nabla \boldsymbol{\theta}. \quad (5.4.88)$$

In that case there are exactly two options for such functions  $\mathbf{S}_1, \mathbf{S}_2$ . They satisfy  $\mathbf{S}_1 \nabla \boldsymbol{\theta} = -\mathbf{S}_2 \nabla \boldsymbol{\theta}$ .

- c) For any two pairs  $(\boldsymbol{\psi}_1, \mathbf{S}_1), (\boldsymbol{\psi}_2, \mathbf{S}_2) \in \mathcal{A}^{\pm F}$  satisfying either  $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2$  or  $\mathbf{S}_1 = \mathbf{S}_2$ , the values

$$q_f(\mathbf{S}_i, \nabla' \mathbf{S}_i) = \tilde{q}_f(\mathbf{b}(\boldsymbol{\psi}) - \mathbf{b}(\boldsymbol{\theta})),$$

for  $i = 1, 2$  and  $j = 1, 2$  are the same, where  $\tilde{q}_f$  is defined in (5.3.8).

*Proof.* In the case a), there is  $\boldsymbol{\psi} \in W^{2,2}(\omega; \mathbb{R}^3)$  such that  $(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^{\pm F}$  iff either  $\nabla \boldsymbol{\psi} = +\mathbf{S} \nabla \boldsymbol{\theta}$  or  $\nabla \boldsymbol{\psi} = -\mathbf{S} \nabla \boldsymbol{\theta}$ . The compatibility condition for these PDEs in terms of  $\boldsymbol{\psi}$  is that the rot of the right hand side is equal to 0, i.e.  $\partial_2(\mathbf{S} \partial_1 \boldsymbol{\theta}) = \partial_1(\mathbf{S} \partial_2 \boldsymbol{\theta})$ . Since  $\boldsymbol{\theta} \in C^2(\omega; \mathbb{R}^3)$  and  $\partial_{12} \boldsymbol{\theta} = \partial_{21} \boldsymbol{\theta}$ , we obtain the first part of the claim a).

In case that compatibility condition is satisfied, each of two PDE systems  $\nabla \boldsymbol{\psi} = \pm \mathbf{S} \nabla \boldsymbol{\theta}$  (regarding the sign  $\pm$ ) admits one solution up to an additive constant vector. It is clear that the sum of two solutions  $\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2$  of different PDEs has gradient equal to 0, so the claim a) is proved.

In the case b), we again start with  $(\psi, \mathbf{S}) \in \mathcal{A}^{\pm F}$  iff  $\nabla \psi = \pm \mathbf{S} \nabla \theta$ . From this we obtain that

$$\nabla \psi^T \nabla \psi = (\pm \mathbf{S} \nabla \theta)^T (\pm \mathbf{S} \nabla \theta) = \nabla \theta^T \nabla \theta.$$

Let us prove the other direction: let  $\nabla \psi^T \nabla \psi = \nabla \theta^T \nabla \theta$  is satisfied. Consequently,

$$\begin{aligned} \begin{bmatrix} \nabla \psi & \mathbf{a}_3^\psi \end{bmatrix}^T \begin{bmatrix} \nabla \psi & \mathbf{a}_3^\psi \end{bmatrix} &= \begin{bmatrix} \nabla \psi^T \nabla \psi & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \nabla \theta^T \nabla \theta & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \nabla \theta & \mathbf{a}_3 \end{bmatrix}^T \begin{bmatrix} \nabla \theta & \mathbf{a}_3 \end{bmatrix}. \end{aligned}$$

That implies that all angles and vector lengths in the basis  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$  are the same as in the basis  $\{\partial_1 \psi, \partial_2 \psi, \mathbf{a}_3^\psi\}$ . For a.e. points  $\mathbf{x}' \in \omega$  let us define a linear operator  $\mathbf{S}$  such that  $\mathbf{S} \begin{bmatrix} \nabla \theta & \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \nabla \psi & \mathbf{a}_3^\psi \end{bmatrix}$ . It is well defined and regular since it maps a basis to a basis. Since all angles and vector lengths are preserved,  $\mathbf{S}$  satisfies

$$\mathbf{S} \begin{bmatrix} \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} & \frac{\mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{a}_1) \mathbf{a}_1}{\|\mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{a}_1) \mathbf{a}_1\|} & \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial_1 \psi}{\|\partial_1 \psi\|} & \frac{\partial_2 \psi - (\partial_2 \psi \cdot \partial_1 \psi) \partial_1 \psi}{\|\partial_2 \psi - (\partial_2 \psi \cdot \partial_1 \psi) \partial_1 \psi\|} & \mathbf{a}_3^\psi \end{bmatrix}$$

meaning that  $\mathbf{S}$  maps an orthonormal basis to another one. Since both basis are right ones, we have  $\mathbf{S} \in \text{SO}(3)$  a.e. The regularity of  $\mathbf{S}$  is a consequence of regularities of  $\psi$  and  $\theta$ .

Also, note that there is exactly one such  $\mathbf{S}_1$  satisfying  $\nabla \psi = \mathbf{S}_1 \nabla \theta$  (since it is uniquely defined on the basis). In the same manner, there is exactly one such  $\mathbf{S}_2$  satisfying  $\nabla \psi = -\mathbf{S}_2 \nabla \theta$ . The rest of the part b) is clear.

In the case c), we have two options. If  $\psi_1 = \psi_2$ , we are in the case b) of this proposition. By Remark 5.3.3 all calculations in Subsection 5.3.2 are same for  $\mathbf{S}_1$  and  $\mathbf{S}_2$  satisfying  $\nabla' \psi = \mathbf{S} \nabla' \theta$ . On the other hand, if  $\mathbf{S}_1 = \mathbf{S}_2$ , we are in the case a). We firstly see that  $\tilde{q}_f$  depends only on gradients of  $\psi$ , so an additive constant vector  $\mathbf{c}$  does not change the value of  $\tilde{q}_f$ . Secondly,  $\tilde{q}_f$  is a quadratic form in  $\nabla' \psi$ , so the  $\pm$  sign does not change its value either.  $\square$

Due to the reasons explained in Subsection 5.3.2 and due to Proposition 5.4.19, in Theorem 5.4.1.c) we obtained the limit which is of the same form as the flexurall shell model appearing in the literature in [40]. In the next proof this is even more clear.

*Proof of Theorem 5.4.2.d).* Let us prove that any  $\psi \in W^{2,2}(\omega; \text{SO}(3))$  with the property that there exists  $\mathbf{S}_0 \in W^{1,2}(\omega; \text{SO}(3))$  such that  $(\psi, \mathbf{S}_0) \in \mathcal{A}^{\pm F}$  is necessarily in the set

of admissible functions for  $\Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right)$ . After applying  $\Gamma$ -limit to the inequality

$$J_{\mathbf{S},h}^{2d,2}(\boldsymbol{\psi}^1) \leq J_h^{2d,2}(\boldsymbol{\psi}^1, \mathbf{S}^1) \quad (5.4.89)$$

to an arbitrary pair  $(\boldsymbol{\psi}^1, \mathbf{S}^1) \in \mathcal{A}^N$ , we obtain

$$\Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right) (\boldsymbol{\psi}^1) \leq J_0^{2d,2}(\boldsymbol{\psi}^1, \mathbf{S}^1). \quad (5.4.90)$$

We plug in  $(\boldsymbol{\psi}, \mathbf{S}_0)$  in last inequality and obtain

$$\Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right) (\boldsymbol{\psi}) \leq J_0^{2d,2}(\boldsymbol{\psi}, \mathbf{S}_0) < +\infty. \quad (5.4.91)$$

This proves the claim from the beginning, and also an inequality we are going to use soon.

Let us now take any admissible  $\boldsymbol{\psi}$  for  $\Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right)$ . We will prove that necessarily there exists  $\mathbf{S}_0 \in L^2(\omega; \text{SO}(3))$  such that  $(\boldsymbol{\psi}, \mathbf{S}_0) \in \mathcal{A}^{\pm F}$ . Since  $\boldsymbol{\psi}$  is admissible, by the definition of  $\Gamma$ -limit (and after taking subsequence if necessary), there exists  $(\boldsymbol{\psi}_h)_{h>0} \subset W^{1,4}(\omega; \mathbb{R}^3)$  which converges to  $\boldsymbol{\psi}$  in  $L^4(\omega; \mathbb{R}^3)$  and such that  $J_{\mathbf{S},h}^{2d,2}(\boldsymbol{\psi}_h) \rightarrow \Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right) (\boldsymbol{\psi})$  and such that for all  $h$  we have uniform bound  $J_{\mathbf{S},h}^{2d,2}(\boldsymbol{\psi}_h) < C$ . By the relation of functionals  $J_{\mathbf{S},h}^{2d,2}$  and  $J_h^{2d,2}$  (through the infimum over  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$ ), there exists  $(\mathbf{S}_h)_{h>0} \subset L^2(\omega; \text{SO}(3))$  such that  $J_h^{2d,2}(\boldsymbol{\psi}_h, \mathbf{S}_h) \leq J_{\mathbf{S},h}^{2d,2}(\boldsymbol{\psi}_h) + h$ . Since the range of functions in this sequence  $(\mathbf{S}_h)_{h>0}$  are rotations, the sequence is trivially uniformly bounded in  $L^2(\omega; \text{SO}(3))$ . Due to Corollary 5.4.16, we obtain that  $\|\mathbf{S}_h\|_{W^{1,2}(\omega; \text{SO}(3))}$  is uniformly bounded, thus there is a subsequence (still denoted by  $h$ ) such that  $\mathbf{S}_h \rightharpoonup \mathbf{S}_0$  in  $W^{1,2}(\omega; \text{SO}(3))$ , for some accumulation point  $\mathbf{S}_0 \in W^{1,2}(\omega; \text{SO}(3))$ . From the compactness theorem, we also have  $\mathbf{S}_h \rightarrow \mathbf{S}_0$  in  $L^2(\omega; \text{SO}(3))$ . By applying the  $\liminf$  to the inequality  $J_h^{2d,2}(\boldsymbol{\psi}_h, \mathbf{S}_h) \leq J_{\mathbf{S},h}^{2d,2}(\boldsymbol{\psi}_h) + h$  and by using the  $\Gamma$ -limit of the family  $(J_h^{2d,2})_{h>0}$  we obtain

$$\begin{aligned} J_0^{2d,2}(\boldsymbol{\psi}, \mathbf{S}_0) &\leq \liminf_{h \rightarrow 0} J_0^{2d,2}(\boldsymbol{\psi}_h, \mathbf{S}_h) \leq \liminf_{h \rightarrow 0} J_{\mathbf{S},0}^{2d,2}(\boldsymbol{\psi}_h) \\ &= \lim_{h \rightarrow 0} J_{\mathbf{S},h}^{2d,2}(\boldsymbol{\psi}_h) = \Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right) (\boldsymbol{\psi}) < +\infty. \end{aligned} \quad (5.4.92)$$

Since  $J_0^{2d,2}(\boldsymbol{\psi}, \mathbf{S}_0) < +\infty$ , this proves that  $(\boldsymbol{\psi}, \mathbf{S}_0) \in \mathcal{A}^{\pm F}$ .

We now have both implications of the claim:  $\boldsymbol{\psi}$  is admissible for  $\Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right)$  iff there exists  $\mathbf{S}_0$  such that  $(\boldsymbol{\psi}, \mathbf{S}_0) \in \mathcal{A}^{\pm F}$ . From Proposition 5.4.19.b) we obtain that this is equivalent to  $\nabla \boldsymbol{\psi}^T \nabla \boldsymbol{\psi} = \nabla \boldsymbol{\theta}^T \nabla \boldsymbol{\theta}$ . Hence we proved that the set of admissible functions

for  $\Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right)$  is exactly  $\mathcal{B}_{\mathbf{S},0}^{2d,2}$ . Also, from inequalities (5.4.91) and (5.4.92) we obtain that for any  $\boldsymbol{\psi} \in \mathcal{B}_{\mathbf{S},0}^{2d,2}$  and its appropriate  $\mathbf{S}_0$  (such that  $(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^{\pm F}$ ) we have

$$\Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right) (\boldsymbol{\psi}) = J_0^{2d,2}(\boldsymbol{\psi}, \mathbf{S}_0) = \frac{1}{12} \int_{\omega} q_f(\mathbf{S}_0, \nabla' \mathbf{S}_0) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}'. \quad (5.4.93)$$

From the definition of  $\tilde{q}_f$  in (5.3.8) and its property (5.3.9) we get that we can write

$$\Gamma \left( (J_{\mathbf{S},h}^{2d,2})_{h>0} \right) (\boldsymbol{\psi}) = \frac{1}{12} \int_{\omega} \tilde{q}_f(\mathbf{b}(\boldsymbol{\psi}) - \mathbf{b}(\boldsymbol{\theta})) \sqrt{a} d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \boldsymbol{\psi} \sqrt{a} d\mathbf{x}' = J_{\mathbf{S},0}^{2d,2}(\boldsymbol{\psi}), \quad (5.4.94)$$

which concludes the proof.  $\square$

#### 5.4.5. Constrained membrane plate model - convergence for the case $\alpha \in \langle 0, 5/3 \rangle$

In this subsection we prove the claims of Theorem 5.4.2.c). The goal is to obtain a limit and compare the result with the constrained membrane plate model obtained as the  $\Gamma$ -limit of the three-dimensional elastic energy in [30], presented in Theorem 3.2.1.c).

Note that there are some additional assumptions:

- the undeformed geometry is flat, i.e. we have  $\boldsymbol{\theta}(\mathbf{x}') = (\mathbf{x}', 0)$ ;
- $\gamma_0 = \emptyset$ , i.e. there is no Dirichlet condition;
- the function  $W_m(\mathbf{Y})$  from the definition of  $w_m$  satisfies

$$W_m(\mathbf{Y}) \leq C \|\mathbf{Y}\|_F^2. \quad (5.4.95)$$

Due to simple geometry assumptions we have  $\mathbf{a}_i = \mathbf{e}_i$  and  $\nabla \boldsymbol{\theta} = \mathbf{I}_{3,2}$ , where

$$\mathbf{I}_{3,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We will again use functions  $\boldsymbol{\Theta}_h$ ,  $\boldsymbol{\Psi}_h$  and  $u_h$  as in Subsection 5.4.4. However, those



definitions and properties are now simpler:

$$\begin{aligned}
\Theta_h(x_1, x_2, x_3) &= (x_1, x_2, hx_3), \\
\Psi_h(x_1, x_2, x_3) &= \psi_h(x_1, x_2) + hx_3 \mathbf{S}_h(x_1, x_2) \mathbf{e}_3, \\
u_h(x_1, x_2, x_3) &= \psi_h(x_1, x_2) + x_3 \mathbf{S}_h(x_1, x_2) \mathbf{e}_3, \\
\nabla u_h \circ \Theta_h &= \begin{bmatrix} \nabla' \psi_h + x_3 \nabla' \mathbf{S}_h \mathbf{e}_3 & \mathbf{S}_h \mathbf{e}_3 \end{bmatrix}.
\end{aligned} \tag{5.4.96}$$

Also, we have  $\Theta(\Omega) = \omega \times [-h/2, h/2]$

We will use a lot of results and ideas from [30]. Firstly, we will use the notation  $O(2, 3)$  from the same paper denoting  $3 \times 2$  matrices with orthonormal columns. After that, we need the following definition.

**Definition 5.4.20** ([30]). An origami map on a bounded open set  $\omega \subset \mathbb{R}^2$  is a pair  $(\mathcal{T}, v)$  where

- $\mathcal{T} = \{T_j\}_j$  is a finite family of (closed, nondegenerate) triangles with disjoint interiors such that  $\omega$  is contained in  $\omega_T := \cup_{T_j \in \mathcal{T}} T_j$ ; each pair  $(T_i, T_j)$  is either disjoint, or shares a corner or shares a side;
- $v \in W^{1,\infty}(\omega_T; \mathbb{R})$  is affine on each  $T_j$ , and  $\nabla' v \in O(2, 3)$  a.e.

Following this definition, for each origami map  $(\mathcal{T}, v)$  authors define degeneracy factor  $\Gamma(\mathcal{T})$ , cardinality  $|\mathcal{T}|$  and maximum jump  $\varphi$  of  $\nabla' v$ . Their precise definitions are not important for further proofs.

They also prove following theorems:

**Theorem 5.4.21** ([30], Theorem 1.8.). Let  $u$  be a short map on a bounded Lipschitz set  $\omega \subset \mathbb{R}^2$ . Then for every  $\delta > 0$  we can find an origami map  $(\mathcal{T}_\delta, v_\delta)$  on  $\omega$  such that

$$\|u - v_\delta\|_{L^\infty(\omega; \mathbb{R}^3)} \leq \delta, \quad |\varphi_\delta| \leq \delta, \tag{5.4.97}$$

where  $\varphi_\delta$  is the maximum jump of  $\nabla' v_\delta$  in  $\omega$ . Furthermore,  $\Gamma(\mathcal{T}_\delta) \leq C_\omega$ .

**Theorem 5.4.22** ([30], Theorem 1.6.). Let  $(\mathcal{T}, v)$  be an origami map on a bounded Lipschitz set  $\omega \subset \mathbb{R}^2$ , and let  $\varphi$  be the maximum jump of  $\nabla' v$ . Then for every  $h > 0$

small enough (depending only on  $\omega$  and  $\mathcal{T}$ ), there is a function  $v_h \in C^2(\omega; \mathbb{R}^3)$  such that

$$\begin{aligned} \|v - v_h\|_{L^\infty(\omega; \mathbb{R}^3)} &\leq C_{\Gamma(\mathcal{T})} \varphi h^{1/3}, \\ \|\nabla' v_h^T \nabla' v_h - \mathbf{I}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + h^2 \|D^2 v_h\|_{L^2(\omega; \mathbb{R}^{3 \times 2 \times 2})}^2 &\leq C_{\Gamma(\mathcal{T})} |\mathcal{T}| \varphi^2 h^{5/3}, \\ \|\text{dist}(\nabla' v_h, O(2, 3))\|_{L^\infty(\omega; \mathbb{R})} + h \|D^2 v_h\|_{L^\infty(\omega; \mathbb{R}^{3 \times 2 \times 2})} &\leq C_{\Gamma(\mathcal{T})} \varphi, \end{aligned} \quad (5.4.98)$$

where  $C_{\Gamma(\mathcal{T})}$  is a non-decreasing function in  $\Gamma(\mathcal{T})$ .

Before proving the main result, we need to prove a lemma on our own.

**Lemma 5.4.23.** There exists  $m > 0$  such that columns  $X_1, X_2$  of all matrices  $\mathbf{X}$  from the set

$$\mathcal{S}_m := \{\mathbf{X} \in \mathbb{M}^{3 \times 2} : \text{dist}(\mathbf{X}, O(2, 3)) \leq m\}$$

satisfy  $\|X_1\| > 0$ ,  $\|X_2\| > 0$ ,  $\|X_1 \times X_2\| \geq \frac{1}{2}$ .

*Proof.* It is clear that  $\mathcal{S}_m$  is compact in  $\mathbb{M}^{3 \times 2}$ . We note that there is a  $m_0 > 0$  small enough such that for all  $m \leq m_0$  in  $\mathcal{S}_m$  all matrices have full rank. Otherwise, as  $m \rightarrow 0$ , there is a sequence  $(\mathbf{X}^m)_{m>0}$  from a compact set whose matrix columns satisfy  $X_1^m \times X_2^m = 0$ . Their accumulation point  $\mathbf{X}^0$  also satisfies  $X_1^0 \times X_2^0 = 0$ , which gives contradiction since  $\text{dist}(\mathbf{X}^m, O(2, 3)) \rightarrow 0$  implies  $\mathbf{X}^0 \in O(2, 3)$ .

Let us take such  $m_0$  and any  $m \leq m_0$ . Since  $\mathcal{S}_m$  is compact and does not contain matrices which are not full rank, the values  $\|X_1\|$ ,  $\|X_2\|$  and  $\|X_1 \times X_2\|$  are strictly greater than zero. Moreover, the function

$$\Phi_1(\mathbf{X}) := \|X_1 \times X_2\| \quad (5.4.99)$$

is a smooth function on the compact set  $\mathcal{S}_m$ , so it is Lipschitz function with constant  $L_{\Phi_1}$ . Thus for all  $\mathbf{X} \in \mathcal{S}_m$  and its closest  $O(2, 3)$  matrix  $\mathbf{R}_\mathbf{X}$  we have

$$\left| \|X_1 \times X_2\| - 1 \right| = \left| \|X_1 \times X_2\| - \|R_1 \times R_2\| \right| \leq L_{\Phi_1} \|\mathbf{X} - \mathbf{R}\| \leq L_{\Phi_1} m. \quad (5.4.100)$$

Thus, with taking  $m$  even smaller if necessary, we can make that for all  $\mathbf{X} \in \mathcal{S}_m$  we have  $\Phi_1(\mathbf{X}) \geq 1/2$ .  $\square$

We are now ready to prove the claim from Theorem 5.4.2.c), by editing proofs of Theorems 1.1. and 1.2. and Lemma 5.1. from the same paper [30].

*Proof of Theorem 5.4.2.c).* Due to nonnegativity of  $w_m$  and  $q_f$  and linearity of the force term, the only thing left to prove in the "lim inf" part is that the only admissible  $\psi$  are the short ones.

Let us take any  $\psi \in L^4(\omega; \mathbb{R}^3)$  and any  $(\psi_h)_{h>0} \subset L^4(\omega; \mathbb{R}^3)$  converging strongly to  $\psi$  in  $L^4(\omega; \mathbb{R}^3)$  with  $\liminf_{h \rightarrow 0} J_{\mathbf{S},h}^{2d,\langle 0,5/3 \rangle}(\psi_h) < +\infty$ . Specially we may conclude (with taking subsequence if necessary) that  $J_{\mathbf{S},h}^{2d,\langle 0,5/3 \rangle}(\psi_h) < C$ . Consequently,  $(\psi_h)_{h>0} \subset W^{1,4}(\omega; \mathbb{R}^3)$ . Let for all  $h > 0$  function  $\mathbf{S}_h \in L^2(\omega; \text{SO}(3))$  denote one for which  $J_h^{2d,\langle 0,5/3 \rangle}(\psi_h, \mathbf{S}_h) \leq J_{\mathbf{S},h}^{2d,\langle 0,5/3 \rangle}(\psi_h) + h$ . Since this implies  $J_h^{2d,\langle 0,5/3 \rangle}(\psi_h, \mathbf{S}_h) < +\infty$ , we conclude  $\mathbf{S}_h \in W^{1,2}(\omega; \text{SO}(3))$  for all  $h > 0$ .

We again use the argument we already used: since rotations  $\text{SO}(3)$  is a bounded set in  $\mathbb{M}^{3 \times 3}$  and  $\mathbf{S}_h$  are measurable functions with  $\mathbf{S}_h(\mathbf{x}') \in \text{SO}(3)$  a.e., we get the bound

$$\|\mathbf{S}_h\|_{L^2(\omega; \text{SO}(3))} \leq M' \quad (5.4.101)$$

for all  $h > 0$  with a constant  $M' > 0$  depending only on the domain  $\omega$ . From coercivity estimate from Corollary 5.4.16 we have

$$h\|\nabla' \mathbf{S}_h\|_{L^2(\omega; \mathbb{R}^{3 \times 6})} \leq Mh^{\alpha/2}, \quad \|\nabla' \psi_h\|_{L^4(\omega; \mathbb{R}^{3 \times 2})} \leq M \quad (5.4.102)$$

and

$$\|\nabla' \psi_h^T \nabla' \psi_h - \mathbf{I}\|_{L^2(\omega; \mathbb{R}^3)}^2 + \|\mathbf{S}_h \mathbf{e}_3 \cdot \partial_1 \psi_h\|_{L^2(\omega; \mathbb{R}^3)}^2 + \|\mathbf{S}_h \mathbf{e}_3 \cdot \partial_2 \psi_h\|_{L^2(\omega; \mathbb{R}^3)}^2 \leq Ch^\alpha. \quad (5.4.103)$$

From these bounds, from Lemma 5.4.18 we have that

$$\frac{1}{h} \int_{\omega \times [-h/2, h/2]} \text{dist}^2(\nabla u_h, \text{SO}(3)) d\mathbf{x} \leq C'h^\alpha, \quad (5.4.104)$$

and

$$\int_{\omega \times [-1/2, 1/2]} \text{dist}^2(\nabla_h \Psi_h, \text{SO}(3)) d\mathbf{x} \leq C'h^\alpha, \quad (5.4.105)$$

by change of variables, with  $u_h$  and  $\Psi_h$  defined in (5.4.96). Moreover, due to definition of  $\Psi_h$  and uniform bounds (5.4.101) and (5.4.102) we have

$$\begin{aligned} \|\nabla \Psi_h\|_{L^2(\omega \times [-1/2, 1/2]; \mathbb{R}^{3 \times 3})}^2 &= \left\| \begin{bmatrix} \nabla' \psi_h + hx_3 \nabla' \mathbf{S}_h \mathbf{e}_3 & \mathbf{S}_h \mathbf{e}_3 \end{bmatrix} \right\|_{L^2(\omega \times [-1/2, 1/2]; \mathbb{R}^{3 \times 3})}^2 \\ &\leq c \left( \|\nabla' \psi_h\|_{L^2(\omega; \mathbb{R}^{3 \times 2})}^2 + h^2 \|\nabla' \mathbf{S}_h\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}^2 + \|\mathbf{S}_h\|_{L^2(\omega; \text{SO}(3))}^2 \right) \leq \tilde{M}. \end{aligned} \quad (5.4.106)$$

Thus, the family  $(\Psi_h)_{h>0}$  is uniformly bounded in  $W^{1,2}(\omega \times [-1/2, 1/2]; \mathbb{R}^3)$ , so there is a convergent subsequence. Let us take any such subsequence (still denoted by  $h$ ) and its limit  $\Psi_0$ .

From (5.4.105) for  $\alpha' = \frac{1}{2}(\alpha + 5/3)$  we obtain

$$\frac{1}{h^{\alpha'}} \int_{\omega \times [-1/2, 1/2]} \text{dist}^2(\nabla_h \Psi_h, \text{SO}(3)) d\mathbf{x} \rightarrow 0. \quad (5.4.107)$$

On the other hand, by Theorem 3.2.1.c) (for  $W_\alpha^1(\mathbf{X}) = \text{dist}^2(\mathbf{X}, \text{SO}(3))$ ,  $\alpha' = \frac{1}{2}(\alpha + 5/3)$ , and forcing equal to 0) and by the "lim inf" part of the definition of  $\Gamma$ -limit for such  $(J_h^{3d, \langle 0, 5/3 \rangle})_{h>0}$  we see that the expression from (5.4.107) tends to 0 if and only if the limit  $\Psi_0$  is a short map. More precisely, we obtain that  $\partial_3 \Psi_0 = 0$ ,  $\nabla \tilde{\Psi}_0^T \nabla \tilde{\Psi}_0 \leq \mathbf{I}$ .

Again due to bound for  $(\mathbf{S}_h)_{h>0}$  in (5.4.102) we conclude that

$$hx_3 \nabla' \mathbf{S}_h \mathbf{e}_3 \rightarrow 0 \text{ in } L^2(\omega \times [-1/2, 1/2]; \mathbb{R}^3)$$

Due to  $\nabla' \psi_h \rightharpoonup \nabla' \psi$  in  $L^4(\omega; \mathbb{R}^3)$  (from compactness and uniform bound (5.4.102)), we have

$$\partial_\beta \Psi_h = \left[ \nabla' \psi_h + hx_3 \nabla' \mathbf{S}_h \mathbf{e}_3 \quad \mathbf{S}_h \mathbf{e}_3 \right] \mathbf{e}_\beta \rightharpoonup \partial_\beta \psi \text{ in } L^2(\Omega^1; \mathbb{R}^3). \quad (5.4.108)$$

By using uniqueness of limits in distributions we get  $\nabla' \tilde{\Psi}_0 = \nabla' \psi$  and  $\nabla' \psi^T \nabla' \psi \leq \mathbf{I}$ , which concludes the proof for the "lim inf" part of Theorem 5.4.2.c).

Let us now proceed to the more complicated part of the proof of the proposition: the "lim sup" part. Without loss of generality we can assume that  $\mathbf{f} = 0$  since it is a linear part of the functional. Let us start from the end of the proof, let us assume that we have obtained a family  $(\psi_h, \mathbf{S}_h) \subset \mathcal{A}^N$  for which bounds

$$h \|\nabla' \mathbf{S}_h\|_{L^2(\omega; \mathbb{R}^{3 \times 6})} \leq M h^{\alpha/2}, \quad \|\nabla' \psi_h\|_{L^4(\omega; \mathbb{R}^{3 \times 2})} \leq M \quad (5.4.109)$$

hold. According to assumptions on  $w_m$  and  $q_f$  (including (5.4.95)) we have

$$\begin{aligned} & h^{-\alpha} \int_\omega w_m(\mathbf{S}_h, \nabla' \psi_h) d\mathbf{x}' + h^{-\alpha+2} \frac{1}{12} \int_\omega q_f(\mathbf{S}_h, \nabla' \mathbf{S}_h) d\mathbf{x}' \\ & \leq C_1 h^{-\alpha} \left( \|\nabla' \psi_h^T \nabla' \psi_h - \mathbf{I}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + \|\nabla' \psi_h^T \mathbf{S}_h \mathbf{I}_{3,2} - \mathbf{I}_{3,2}^T \mathbf{S}_h^T \nabla' \psi_h\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 \right. \\ & \quad \left. + \|\mathbf{S}_h \mathbf{e}_3 \cdot \partial_1 \psi_h\|_{L^2(\omega; \mathbb{R})}^2 + \|\mathbf{S}_h \mathbf{e}_3 \cdot \partial_2 \psi_h\|_{L^2(\omega; \mathbb{R})}^2 \right) \\ & \quad + C_2 h^{-\alpha+2} (1 + \|\nabla' \mathbf{S}_h\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}^2) \end{aligned}$$

Using Lemma 5.4.18 (due to (5.4.109)) we obtain

$$\begin{aligned} h^{-\alpha} \int_{\omega} w_m(\mathbf{S}_h, \nabla' \psi_h) d\mathbf{x}' + h^{-\alpha+2} \frac{1}{12} \int_{\omega} q_f(\mathbf{S}_h, \nabla' \mathbf{S}_h) d\mathbf{x}' \\ \leq C_3 \cdot \mathcal{D}(h; \psi_h, \mathbf{S}_h) + \mathcal{O}(h^{5/3-\alpha}) \end{aligned} \quad (5.4.110)$$

with

$$\begin{aligned} \mathcal{D}(h; \psi_h, \mathbf{S}_h) &:= h^{-\alpha-1} \int_{\omega \times [h/2, -h/2]} \text{dist}^2(\nabla u_h, \text{SO}(3)) \\ &+ h^{-\alpha} \|\nabla' \psi_h^T \mathbf{S}_h \mathbf{I}_{3,2} - \mathbf{I}_{3,2}^T \mathbf{S}_h^T \nabla' \psi_h\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + h^{-\alpha+2} \|\nabla' \mathbf{S}_h\|_{L^2(\omega; \mathbb{R}^{6})}^2 \end{aligned} \quad (5.4.111)$$

and  $u_h : \omega \times [-\frac{h}{2}, \frac{h}{2}]$ ,  $u_h(\mathbf{x}) := \psi(\mathbf{x}') + x_3 \mathbf{S}(\mathbf{x}') e_3$ . We will prove that for arbitrary short  $\psi \in W^{1,4}(\omega; \mathbb{R}^3)$  there exists  $(\psi_{n(h)}, \mathbf{S}_{n(h)})_{h>0}$  such that  $\psi_{n(h)} \rightarrow \psi$  in  $L^4(\omega; \mathbb{R}^3)$  and  $\mathcal{D}(h; \psi_{n(h)}, \mathbf{S}_{n(h)}) \rightarrow 0$  as  $h \rightarrow 0$ .

So, let us take any short  $\psi \in W^{1,4}(\omega; \mathbb{R}^3)$ . Note that this implies  $\psi \in W^{1,\infty}(\omega; \mathbb{R}^3)$ . For  $\psi$  due to Theorem 5.4.21 for every  $\delta > 0$  there is an origami map  $(\mathcal{T}_\delta, \psi_\delta)$  which is  $\delta$  close to  $\psi$  in  $L^\infty(\omega; \mathbb{R}^3)$  with  $|\varphi_\delta| \leq \delta$ . Let us take sequence  $h_k \rightarrow 0$ . For  $k \geq k_0(\delta)$  big enough we can apply Theorem 5.4.22 to find  $\psi_{\delta,k} \in C^2(\omega; \mathbb{R}^3)$  satisfying bounds

$$\begin{aligned} \|\psi_\delta - \psi_{\delta,k}\|_{L^\infty(\omega; \mathbb{R}^3)} &\leq C_{\Gamma(\mathcal{T}_\delta)} \varphi_\delta h_k^{1/3}, \\ \|\nabla' \psi_{\delta,k}^T \nabla' \psi_{\delta,k} - \mathbf{I}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + h_k^2 \|D^2 \psi_{\delta,k}\|_{L^2(\omega; \mathbb{R}^{3 \times 2 \times 2})}^2 &\leq C_{\Gamma(\mathcal{T}_\delta)} |\mathcal{T}_\delta| \varphi_\delta^2 h_k^{5/3}, \\ \|\text{dist}(\nabla' \psi_{\delta,k}, O(2, 3))\|_{L^\infty(\omega; \mathbb{R})} + h_k \|D^2 \psi_{\delta,k}\|_{L^\infty(\omega; \mathbb{R}^{3 \times 2 \times 2})} &\leq C_{\Gamma(\mathcal{T}_\delta)} \varphi_\delta, \end{aligned} \quad (5.4.112)$$

Since constant  $C_{\Gamma(\mathcal{T}_\delta)}$  is a non-decreasing function in  $\Gamma(\mathcal{T}_\delta)$ , and  $\Gamma(\mathcal{T}_\delta) \leq C_\omega$ , we have  $C_{\Gamma(\mathcal{T}_\delta)} \leq C'_\omega$ . Let us focus only on deltas that are so small such that  $\delta C'_\omega \leq m$ , where  $m$  is from Lemma 5.4.23. Specially, we have that

$$\|\text{dist}(\nabla' \psi_{\delta,k}, O(2, 3))\|_{L^\infty(\omega; \mathbb{R})} \leq C_{\Gamma(\mathcal{T}_\delta)} \varphi_\delta \leq \delta C'_\omega \leq m, \quad (5.4.113)$$

so  $\nabla' \psi_{\delta,k} \in \mathcal{S}_m$ . Also note that from the triangle inequality together with the first bound in (5.4.112) we have

$$\|\psi_{\delta,k} - \psi\|_{L^\infty(\omega; \mathbb{R}^3)} \leq \|\psi_{\delta,k} - \psi_\delta\|_{L^\infty(\omega; \mathbb{R}^3)} + \|\psi_\delta - \psi\|_{L^\infty(\omega; \mathbb{R}^3)} \leq C_{\Gamma(\mathcal{T}_\delta)} \varphi_\delta h_k^{1/3} + \delta. \quad (5.4.114)$$

Let us define

$$\Phi_2(\mathbf{X}) = \left[ \frac{X_1}{\|X_1\|} \quad \frac{X_1 \times X_2}{\|X_1 \times X_2\|} \times \frac{X_1}{\|X_1\|} \quad \frac{X_1 \times X_2}{\|X_1 \times X_2\|} \right] \quad (5.4.115)$$

and  $\mathbf{S}_{\delta,k} := \Phi_2(\nabla' \boldsymbol{\psi}_{\delta,k})$ . Note that  $\Phi_2(\mathbf{X}) \in \text{SO}(3)$  for all  $\mathbf{X}$  for which the function  $\Phi_2$  is defined. On the domain  $\mathcal{S}_m$  the function  $\Phi_2$  is well-defined, and moreover it is smooth. Since  $\boldsymbol{\psi}_{\delta,k}$  are  $C^2$  functions, and  $\nabla' \boldsymbol{\psi}_{\delta,k} \in \mathcal{S}_m$ , we conclude that  $\mathbf{S}_{\delta,k}$  are in  $C^1(\omega; \text{SO}(3))$ .

Note that  $\nabla' \mathbf{S}_{\delta,k} = \nabla \Phi_2(\nabla' \boldsymbol{\psi}_{\delta,k}) (D^2 \boldsymbol{\psi}_{\delta,k})$ . Since  $\Phi_2$  is smooth on the compact set  $\mathcal{S}_m$ , its derivative is bounded from above with a constant  $M_{\Phi'_2}$ , thus for all  $\mathbf{x}' \in \omega$  we obtain

$$\|\nabla' \mathbf{S}_{\delta,k}(\mathbf{x}')\|_F^2 \leq M_{\Phi'_2}^2 \|D^2 \boldsymbol{\psi}_{\delta,k}(\mathbf{x}')\|_F^2 \quad (5.4.116)$$

(the Frobenius matrix norm is submultiplicative with a constant depending on the dimensions of matrices). From here and from second bound in (5.4.112) we obtain that there is a constant  $C_S$  such that

$$\|\nabla' \mathbf{S}_{\delta,k}\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}^2 \leq C_S \cdot C_{\Gamma(\mathcal{T}_\delta)} |\mathcal{T}_\delta| \varphi_\delta^2 h_k^{5/3}. \quad (5.4.117)$$

By noting that in each  $\mathbf{x}' \in \omega$  the matrix  $\nabla' \boldsymbol{\psi}_{\delta,k}^T \mathbf{S}_{\delta,k} \mathbf{I}_{3,2} - \mathbf{I}_{3,2}^T \mathbf{S}_{\delta,k}^T \nabla' \boldsymbol{\psi}_{\delta,k}$  is a skew-symmetric  $2 \times 2$  matrix, we obtain that

$$\|\nabla' \boldsymbol{\psi}_{\delta,k}^T \mathbf{S}_{\delta,k} \mathbf{I}_{3,2} - \mathbf{I}_{3,2}^T \mathbf{S}_{\delta,k}^T \nabla' \boldsymbol{\psi}_{\delta,k}\|_F^2 = 2 \left| \partial_1 \boldsymbol{\psi}_{\delta,k} \cdot \mathbf{S}_{\delta,k} \mathbf{e}_2 - \partial_2 \boldsymbol{\psi}_{\delta,k} \cdot \mathbf{S}_{\delta,k} \mathbf{e}_1 \right|^2. \quad (5.4.118)$$

Let us define

$$\Phi_3(\mathbf{X}) = \left| X_1 \cdot \Phi_2(\mathbf{X}) \mathbf{e}_2 - X_2 \cdot \Phi_2(\mathbf{X}) \mathbf{e}_1 \right|. \quad (5.4.119)$$

We note that  $\Phi_3(\nabla' \boldsymbol{\psi}_{\delta,k}) = \left| \partial_1 \boldsymbol{\psi}_{\delta,k} \cdot \mathbf{S}_{\delta,k} \mathbf{e}_2 - \partial_2 \boldsymbol{\psi}_{\delta,k} \cdot \mathbf{S}_{\delta,k} \mathbf{e}_1 \right|^2$ . Also,  $\Phi_3$  is a smooth function on a compact set  $\mathcal{S}_m$ , so specially it is a Lipschitz function with a constant  $L_{\Phi_3}$ . Let  $\mathbf{X} \in \mathcal{S}_m$  and  $\mathbf{R}_\mathbf{X}$  from  $O(2, 3)$  be its closest matrix. Since  $\Phi_3 = 0$  on  $O(2, 3)$  we have

$$\Phi_3(\mathbf{X})^2 = |\Phi_3(\mathbf{X}) - \Phi_3(\mathbf{R}_\mathbf{X})|^2 \leq L_{\Phi_3}^2 \text{dist}^2(\mathbf{X}, O(2, 3)) \quad (5.4.120)$$

and consequently

$$\|\nabla' \boldsymbol{\psi}_{\delta,k}^T \mathbf{S}_{\delta,k} \mathbf{I}_{3,2} - \mathbf{I}_{3,2}^T \mathbf{S}_{\delta,k}^T \nabla' \boldsymbol{\psi}_{\delta,k}\|_F^2 \leq 2L_{\Phi_3}^2 \text{dist}^2(\nabla' \boldsymbol{\psi}_{\delta,k}, O(2, 3)). \quad (5.4.121)$$

Let us define  $u_{\delta,k} : \omega \times [-\frac{h_k}{2}, \frac{h_k}{2}]$ ,  $u_{\delta,k}(\mathbf{x}) := \boldsymbol{\psi}_{\delta,k}(\mathbf{x}') + x_3 \mathbf{S}_{\delta,k}(\mathbf{x}') \mathbf{e}_3$ . Let us fix  $\mathbf{x}' \in \omega$ , and let  $\begin{bmatrix} R_1 & R_2 \end{bmatrix}$  be a  $O(2, 3)$  matrix which is the closest to  $\nabla' \boldsymbol{\psi}_{\delta,k}(\mathbf{x}')$ . Let  $\mathbf{R} \in \text{SO}(3)$  be with first two columns equal to  $R_1, R_2$ . In other words,  $R_3 := \mathbf{R} \mathbf{e}_3 = \frac{R_1 \times R_2}{\|R_1 \times R_2\|}$ . Since  $\nabla u_{\delta,k} = \begin{bmatrix} \nabla' \boldsymbol{\psi}_{\delta,k} + x_3 \nabla' \mathbf{S}_{\delta,k} \mathbf{e}_3 & \mathbf{S}_{\delta,k} \mathbf{e}_3 \end{bmatrix}$ , we can bound

$$\begin{aligned} \text{dist}^2(\nabla u_{\delta,k}, \text{SO}(3)) &\leq \|\nabla u_{\delta,k} - \mathbf{R}\|_F^2 \\ &\leq 2 \left( \|\nabla' \boldsymbol{\psi}_{\delta,k} - \begin{bmatrix} R_1 & R_2 \end{bmatrix}\|_F^2 + |x_3|^2 \|\nabla' \mathbf{S}_{\delta,k}\|_F^2 + \|\mathbf{S}_{\delta,k} \mathbf{e}_3 - R_3\|_F^2 \right). \end{aligned} \quad (5.4.122)$$

Due to (5.4.116),  $|x_3|^2 \|\nabla' \mathbf{S}_{\delta,k}\|_F^2 \leq h_k^2 M_{\Phi_2}'^2 \|D^2 \boldsymbol{\psi}_{\delta,k}\|_F$ . Since  $\Phi_1(\mathbf{X}) = \|X_1 \times X_2\|$  is a Lipschitz function on  $\mathcal{S}_m$  with the constant  $L_{\Phi_1}$ , the last term in (5.4.122) can be bound by  $L_{\Phi_1}^2 \|\nabla' \boldsymbol{\psi}_{\delta,k} - [R_1 \ R_2]\|_F^2$ . Altogether, there is a constant  $C_u$  such that

$$\text{dist}^2(\nabla u_{\delta,k}, \text{SO}(3)) \leq C_u (\text{dist}^2(\nabla' \boldsymbol{\psi}_{\delta,k}, O(2,3)) + h_k^2 \|D^2 \boldsymbol{\psi}_{\delta,k}\|_F). \quad (5.4.123)$$

Note that now we have bounded all expressions in  $\mathcal{D}(h; \boldsymbol{\psi}_{\delta,k}, \mathbf{S}_{\delta,k})$  from (5.4.111). Indeed, by using (5.4.123), (5.4.121), and (5.4.117) we obtain

$$\mathcal{D}(h_k; \boldsymbol{\psi}_{\delta,k}, \mathbf{S}_{\delta,k}) \leq C h_k^{-\alpha} \left( \int_{\omega} \text{dist}^2(\nabla' \boldsymbol{\psi}_{\delta,k}, O(2,3)) + h_k^2 \|D^2 \boldsymbol{\psi}_{\delta,k}\|_{L^2(\omega; \mathbb{R}^{3 \times 2 \times 2})}^2 \right). \quad (5.4.124)$$

where  $C$  is independent of  $\delta$  and  $k$ . By second inequality in (5.4.112) and Lemma 5.5.9 we bound

$$\mathcal{D}(h_k; \boldsymbol{\psi}_{\delta,k}, \mathbf{S}_{\delta,k}) \leq C \cdot C_{\Gamma_{\mathcal{T}_{\delta}}} |\mathcal{T}_{\delta}| \varphi_{\delta}^2 h_k^{5/3-\alpha}. \quad (5.4.125)$$

To finish the proof, we have to perform carefully the diagonalisation procedure for the family  $(\boldsymbol{\psi}_{\delta,k}, \mathbf{S}_{\delta,k})_{(\delta,k)}$  such that we can use the last bound in the inequality (5.4.110). Let us for arbitrary small  $\delta > 0$  define  $k(\delta)$  sufficiently large such that

$$C_{\Gamma_{\mathcal{T}_{\delta}}} |\mathcal{T}_{\delta}| \varphi_{\delta}^2 h_{k(\delta)}^{5/3-\alpha} \leq \delta \text{ and } C_{\Gamma(\mathcal{T}_{\delta})} \varphi_{\delta} h_{k(\delta)}^{1/3} \leq \delta, \quad (5.4.126)$$

(which is possible since  $(h_k)_{k>0}$  are independent of  $\delta$  and converge to 0) and let us define family  $(\boldsymbol{\psi}_{\delta}, \mathbf{S}_{\delta})_{\delta>0}$  by  $(\boldsymbol{\psi}_{\delta}, \mathbf{S}_{\delta}) = (\boldsymbol{\psi}_{\delta,k(\delta)}, \mathbf{S}_{\delta,k(\delta)})$ .

By combining (5.4.126) and (5.4.117), we conclude that we have uniform bound (in terms of  $\delta$ ) of the form  $h_{k(\delta)} \|\nabla' \mathbf{S}_{\delta}\|_{L^2(\omega; \mathbb{R}^{3 \times 6})} \leq C_{\mathbf{S}}^{1/2} h_{k(\delta)}^{\alpha/2}$ . Since we already concluded that for all  $\delta$  and  $k$  we have  $\nabla' \boldsymbol{\psi}_{\delta,k} \in \mathcal{S}_m$ , which is a compact set in  $\mathbb{M}^{3 \times 2}$ , we trivially have a uniform bound for  $\|\nabla' \boldsymbol{\psi}_{\delta}\|_{L^4(\omega; \mathbb{R}^{3 \times 2})}$ . That means that (5.4.109) is satisfied and we are allowed to use (5.4.110) to obtain

$$\begin{aligned} & h_{k(\delta)}^{-\alpha} \int_{\omega} w_m(\mathbf{S}_{\delta}, \nabla' \boldsymbol{\psi}_{\delta}) d\mathbf{x}' + h_{k(\delta)}^{-\alpha+2} \frac{1}{12} \int_{\omega} q_f(\mathbf{S}_{\delta}, \nabla' \mathbf{S}_{\delta}) d\mathbf{x}' \\ & \leq C_3 \cdot \mathcal{D}(h_{k(\delta)}; \boldsymbol{\psi}_{\delta}, \mathbf{S}_{\delta}) + \mathcal{O}(h_{k(\delta)}^{5/3-\alpha}). \end{aligned} \quad (5.4.127)$$

By (5.4.125) and first assumption from (5.4.126), we get

$$h_{k(\delta)}^{-\alpha} \int_{\omega} w_m(\mathbf{S}_{\delta}, \nabla' \boldsymbol{\psi}_{\delta}) d\mathbf{x}' + h_{k(\delta)}^{-\alpha+2} \frac{1}{12} \int_{\omega} q_f(\mathbf{S}_{\delta}, \nabla' \mathbf{S}_{\delta}) d\mathbf{x}' \leq C' \delta + \mathcal{O}(h_{k(\delta)}^{5/3-\alpha}) \rightarrow 0 \quad (5.4.128)$$

as  $\delta \rightarrow 0$ . Finally, from the second assumption in (5.4.126) and (5.4.114), we obtain that  $\psi_\delta$  converges to  $\psi$  in  $L^\infty(\omega; \mathbb{R}^3)$ , and consequently in  $L^4(\omega; \mathbb{R}^3)$  as well, so this finally completes the "lim sup" part of the proof.  $\square$

#### 5.4.6. Convergence for cases $\alpha \in \langle -1, 0 \rangle$ and $\alpha \in \langle 2, +\infty \rangle$

For completeness of the analysis, we will establish the  $\Gamma$ -limit of the family of functionals  $(J_h^{2d,\alpha})_{h>0}$  in two simpler cases:  $\alpha \in \langle -1, 0 \rangle$  and  $\alpha \in \langle 2, +\infty \rangle$ . After this analysis the only case for which we didn't find the  $\Gamma$ -limit is  $\alpha \in [5/3, 2]$ , which does not exist in the analysis of 3d equations as well.

We start with a lemma for the case  $\alpha \in \langle -1, 0 \rangle$ .

**Lemma 5.4.24.** Let  $\omega$  be a polygonal domain and  $\gamma_0 = \emptyset$  (i.e. there is no Dirichlet boundary condition). Let  $(\psi, \mathbf{S}) \in L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$ . Then there exists a sequence  $(\psi_n, \mathbf{S}_n)_{n \geq 1} \subset \mathcal{A}^N$  such that  $(\psi_n, \mathbf{S}_n) \rightarrow (\psi, \mathbf{S})$  strongly in  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$ .

*Proof.* In the view of Lemma A.1 with Remark A.2 (and the diagonalization argument), it is sufficient to prove that there is a sequence  $(\psi_n, \mathbf{S}_n)_{n \geq 1} \subset \mathcal{A}^M$  such that  $(\psi_n, \mathbf{S}_n) \rightarrow (\psi, \mathbf{S})$  strongly in  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$ , where  $\mathcal{A}^M$  is defined in (5.4.6). Let us take arbitrary  $n \geq 1$ . Due to Theorem 4.4.20. from [17] (with taking smooth approximation before if necessary), there is a triangulation  $\mathcal{T}_n$  of  $\omega$  and piecewise constant functions  $\phi_n$  (with respect to  $\mathcal{T}_n$ ) such that  $\|\phi_n - \psi\|_{L^4(\omega; \mathbb{R}^3)} < \frac{1}{n}$ . On each triangle  $T$  of the triangulation we take a smaller triangle  $T'$  similar to  $T$  with the incenter of  $T$  as center of homothety and coefficient of homothety equal to  $1 - \delta$ . Now we define  $\psi_n$  on each triangle in the following manner:  $\psi_n = \phi_n$  in  $T'$ ,  $\psi_n = 0$  on the boundary of  $T$ , and piecewise affine on  $T \setminus T'$  (with respect to three trapezoids in  $T \setminus T'$ ). It is clear that as  $\delta \rightarrow 0$  we have  $\psi_n \rightarrow \phi_n$  in  $L^4(\omega; \mathbb{R}^3)$ , so we can choose  $\delta = \delta(n)$  such that  $\|\phi_n - \psi_n\|_{L^4(\omega; \mathbb{R}^3)} < \frac{1}{n}$ . By the triangle inequality,  $\psi_n \rightarrow \psi$  in  $L^4(\omega; \mathbb{R}^3)$ . Also,  $\psi_n \in W^{1,4}(\omega; \mathbb{R}^3)$  since it is piecewise affine. Also note that on  $T'$  the function  $\psi_n$  is constant and on  $T \setminus T'$  is constant along one direction (parallel to sides of  $T$ ). This is why  $\partial_1 \psi_n \times \partial_2 \psi_n = 0$ , so for the choice  $\mathbf{S}_n := \mathbf{S}$  we will have  $\det \begin{bmatrix} \mathbf{S}_n \mathbf{a}_3 & \partial_1 \psi_n & \partial_2 \psi_n \end{bmatrix} = 0$ . Thus the pair  $(\psi_n, \mathbf{S}_n) \in \mathcal{A}^M$  is arbitrarily close to  $(\psi, \mathbf{S})$ , which completes the proof.  $\square$



**Remark 5.4.25.** Note that if  $\psi$  is more regular ( $\psi \in W^{1,4}(\omega; \mathbb{R}^3)$ ), we don't need the construction of the sequence from the main part of Lemma 5.4.24, we can use only the sequence from Lemma A.1.

*Proof of Theorem 5.4.1.a).* Since the first two integrals in the definition of functional  $J_h^{2d, \langle -1, 0 \rangle}(\psi, \mathbf{S})$  are nonnegative and the last is linear, the "lim inf" part is clear.

For the "lim sup" part, we can neglect the force term since it is linear. Let us take any  $(\psi, \mathbf{S}) \in L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$ . Let us take any sequence  $(\psi_n, \mathbf{S}_n)$  from Lemma 5.4.24 that converges strongly to  $(\psi, \mathbf{S})$  in  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  and let us define

$$n(h) := \arg \max_{n \geq 1} \{ \|\nabla' \psi_n\|_{L^4(\omega; \mathbb{R}^{3 \times 3})}^4 + \|\nabla' \mathbf{S}_n\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}^2 < h^{\alpha/2} \}, \quad (5.4.129)$$

in case that  $(s_n)_{n \geq 1}$ ,

$$s_n := \|\nabla' \psi_n\|_{L^4(\omega; \mathbb{R}^{3 \times 3})}^4 + \|\nabla' \mathbf{S}_n\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}^2,$$

is not uniformly bounded (note that  $h^{\alpha/2} \rightarrow +\infty$  as  $h \rightarrow 0$ , so the mapping is well defined for all sufficiently small values of  $h$ ), and  $n(h) = \lceil 1/h \rceil$  if  $(s_n)_{n \geq 1}$  is uniformly bounded.

If  $(s_n)_{n \geq 1}$  is uniformly bounded, then clearly  $n(h) \rightarrow +\infty$  and due to  $h^{\alpha/2} \rightarrow +\infty$  the bound  $s_n < h^{\alpha/2}$  is satisfied for sufficiently small values of  $h$ . If  $(s_n)_{n \geq 1}$  is not uniformly bounded, as  $h \rightarrow 0$ , due to  $h^{\alpha/2} \rightarrow +\infty$  we have  $n(h) \rightarrow +\infty$ . In both cases we have that  $(\psi_h, \mathbf{S}_h)_{h>0}$ , defined as  $(\psi_h, \mathbf{S}_h) := (\psi_{n(h)}, \mathbf{S}_{n(h)})$ , converges to  $(\psi, \mathbf{S})$  strongly in  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$ .

Due to the growth property in Lemma 5.4.4 for sufficiently small  $h$  we have

$$\begin{aligned} J_h^{2d, \langle -1, 0 \rangle}(\psi_h, \mathbf{S}_h) &\leq h^{-\alpha} C(1 + \|\nabla' \psi_h\|_{L^4(\omega; \mathbb{R}^{3 \times 3})}^4) + h^{-\alpha+2} C(1 + \|\nabla' \mathbf{S}_h\|_{L^2(\omega; \mathbb{R}^{3 \times 6})}^2) \\ &\leq C(h^{-\alpha} + h^{-\alpha/2} + h^{-\alpha/2+2}) \rightarrow 0. \end{aligned}$$

This proves that  $\limsup J_h^{2d, \langle -1, 0 \rangle}(\psi_h, \mathbf{S}_h) = 0 = J_0^{2d, \langle -1, 0 \rangle}(\psi, \mathbf{S})$ .  $\square$

*Proof of Theorem 5.4.2.a).* Directly from the inequality

$$-\int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}' \leq J_{\mathbf{S}, h}^{2d, \langle -1, 0 \rangle}(\psi) \leq J_h^{2d, \langle -1, 0 \rangle}(\psi, \mathbf{S})$$

and applying the  $\Gamma$ -limit.  $\square$

Let us now prove the results for the case  $\alpha \in \langle 2, +\infty \rangle$ .

*Proof of Theorem 5.4.1.d).* Let us prove the conditions in the set of admissible deformations. Let  $(J_h^{2d,2})_{h>0}$ ,  $J_0^{2d,2}$ , and  $\mathcal{A}^{\pm F}$  be the energy functionals for  $h > 0$ , its  $\Gamma$ -limit and the set of admissible functions from Theorem 5.4.1.c). Since now  $\alpha > 2$ , we have  $J_h^{2d,\langle 2,+\infty \rangle}(\psi, \mathbf{S}) \geq J_h^{2d,2}(\psi, \mathbf{S})$ , so the same holds for their  $\Gamma$ -limits and the set of admissible functions of  $\Gamma$ -limit of  $(J_h^{2d,\langle 2,+\infty \rangle})_{h>0}$  is necessarily a subset of  $\mathcal{A}^{\pm F}$ . Now, without the loss of generality let  $\mathbf{f} = 0$  (since the linear part will not affect the  $\Gamma$ -limit). Note that now all functionals are nonnegative.

Let us take any  $(\psi, \mathbf{S}) \in \mathcal{A}^{\pm F}$  and any  $(\psi_h, \mathbf{S}_h)_{h>0} \subset W^{1,4}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3))$  that strongly converges to  $(\psi, \mathbf{S})$  in  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  and such that the family  $J_h^{2d,\langle 2,+\infty \rangle}(\psi_h, \mathbf{S}_h)$  is uniformly bounded by  $C > 0$ . If there does not exist such sequence, there is nothing to prove, the "lim inf" inequality is trivially satisfied. For those functions we have  $\liminf_{h \rightarrow 0} J_h^{2d,2}(\psi_h, \mathbf{S}_h) \geq J_0^{2d,2}(\psi, \mathbf{S})$ , so there exists a subsequence of  $(\psi_h, \mathbf{S}_h)_{h>0}$  (still denoted the same) such that

$$(\forall \delta > 0) (\exists h_0 > 0) (\forall h \in \langle 0, h_0 \rangle) J_h^{2d,2}(\psi_h, \mathbf{S}_h) \geq J_0^{2d,2}(\psi, \mathbf{S}) - \delta. \quad (5.4.130)$$

Let us take  $\delta = \frac{1}{2} J_0^{2d,2}(\psi, \mathbf{S})$  and multiply the last inequality by  $h^{2-\alpha}$ . Then we have

$$C \geq J_h^{2d,\langle 2,+\infty \rangle}(\psi_h, \mathbf{S}_h) \geq \frac{1}{2h^{\alpha-2}} J_0^{2d,2}(\psi, \mathbf{S}), \quad (5.4.131)$$

so when  $h \rightarrow 0$  we obtain that necessarily  $J_0^{2d,2}(\psi, \mathbf{S}) = 0$ . From coercivity of  $q_f$  we obtain  $\nabla' \mathbf{S} = 0$ , thus  $\mathbf{S}$  is a constant  $\text{SO}(3)$  matrix. Now the conditions in the set of admissible functions  $\mathcal{B}_0^{2d,2}$  is a consequence of  $\mathbf{S}$  being a constant matrix and conditions in  $\mathcal{A}^{\pm F}$ .

If the Dirichlet boundary condition is prescribed on  $\gamma_0$  (with positive capacity), then firstly from boundary condition for  $\mathbf{S}$  we obtain  $\mathbf{S} = \mathbf{I}$ . Secondly, from  $\mathcal{A}^{\pm F} = \mathcal{A}^F$  (from Theorem 5.4.1.c)) we obtain  $\psi = \theta + \mathbf{d}$ . Thirdly, from the boundary condition for  $\psi$  we obtain that it is necessarily  $\mathbf{d} = 0$ , i.e.  $\psi = \theta$ .

For the "lim sup" part of the proof, for arbitrary  $(\psi, \mathbf{S}) \in \mathcal{B}_0^{2d,\langle 2,+\infty \rangle}$  we take the constant sequence  $(\psi_h, \mathbf{S}_h)_{h>0}$ ,  $(\psi_h, \mathbf{S}_h) = (\psi, \mathbf{S})$ . Since for such functions  $w_m(\mathbf{S}, \nabla' \psi) = q_f(\mathbf{S}, \nabla' \mathbf{S}) = 0$ , the proof is finished.  $\square$

*Proof of Theorem 5.4.2.e).* We approach the same as in the proof of Theorem 5.4.2.d).

Let us prove that any  $\psi \in C(\omega; \mathbb{R}^3)$  with the property that there exists  $\mathbf{S}_0 \in \text{SO}(3)$  such that  $(\psi, \mathbf{S}_0) \in \mathcal{B}_0^{2d, \langle 2, +\infty \rangle}$  is necessarily in the set of admissible functions for the functional  $\Gamma \left( (J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle})_{h>0} \right)$ . After applying  $\Gamma$ -limit to the inequality

$$J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle}(\psi^1) \leq J_h^{2d, \langle 2, +\infty \rangle}(\psi^1, \mathbf{S}^1) \quad (5.4.132)$$

to an arbitrary pair  $(\psi^1, \mathbf{S}^1) \in \mathcal{A}^N$ , we obtain

$$\Gamma \left( (J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle})_{h>0} \right) (\psi^1) \leq J_0^{2d, \langle 2, +\infty \rangle}(\psi^1, \mathbf{S}^1). \quad (5.4.133)$$

We plug in  $(\psi, \mathbf{S}_0)$  in last inequality and obtain

$$\Gamma \left( (J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle})_{h>0} \right) (\psi) \leq J_0^{2d, \langle 2, +\infty \rangle}(\psi, \mathbf{S}_0) < +\infty. \quad (5.4.134)$$

This proves the claim from the beginning, and also an inequality we are going to use soon.

Let us now take any  $\psi$  admissible for  $\Gamma \left( (J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle})_{h>0} \right)$ . We will prove that necessarily there exists  $\mathbf{S}_0 \in \text{SO}(3)$  such that  $(\psi, \mathbf{S}_0) \in \mathcal{B}_0^{2d, \langle 2, +\infty \rangle}$ . Since  $\psi$  is admissible, by the definition of  $\Gamma$ -limit (and after taking subsequence if necessary), there exists  $(\psi_h)_{h>0} \subset W^{1,4}(\omega; \mathbb{R}^3)$  which converges to  $\psi$  in  $L^4(\omega; \mathbb{R}^3)$  and such that we have  $J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle}(\psi_h) \rightarrow \Gamma \left( (J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle})_{h>0} \right) (\psi)$  and such that for all  $h$  we have uniform bound  $J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle}(\psi_h) < C$ . By the relation of functionals  $J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle}$  and  $J_h^{2d, \langle 2, +\infty \rangle}$  (through the infimum over  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$ ), there exists  $(\mathbf{S}_h)_{h>0} \subset L^2(\omega; \text{SO}(3))$  such that  $J_h^{2d, \langle 2, +\infty \rangle}(\psi_h, \mathbf{S}_h) \leq J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle}(\psi_h) + h$ . Since the range of functions in this sequence  $(\mathbf{S}_h)_{h>0}$  are rotations, the sequence is trivially uniformly bounded in  $L^2(\omega; \text{SO}(3))$ . Due to Corollary 5.4.16 for  $\alpha > 2$ , we obtain that  $\|\mathbf{S}_h\|_{W^{1,2}(\omega; \text{SO}(3))}$  is uniformly bounded, thus there is a subsequence (still denoted by  $h$ ) such that  $\mathbf{S}_h \rightharpoonup \mathbf{S}_0$  in  $W^{1,2}(\omega; \text{SO}(3))$ , for some accumulation point  $\mathbf{S}_0 \in W^{1,2}(\omega; \text{SO}(3))$ . From the compactness theorem, we also have  $\mathbf{S}_h \rightarrow \mathbf{S}_0$  in  $L^2(\omega; \text{SO}(3))$ . By applying the  $\liminf$  to the inequality  $J_h^{2d, \langle 2, +\infty \rangle}(\psi_h, \mathbf{S}_h) \leq J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle}(\psi_h) + h$  and by using the  $\Gamma$ -limit of the family  $(J_h^{2d, \langle 2, +\infty \rangle})_{h>0}$  we obtain

$$\begin{aligned} J_0^{2d, \langle 2, +\infty \rangle}(\psi, \mathbf{S}_0) &\leq \liminf_{h \rightarrow 0} J_0^{2d, \langle 2, +\infty \rangle}(\psi_h, \mathbf{S}_h) \leq \liminf_{h \rightarrow 0} J_{\mathbf{S}_0}^{2d, \langle 2, +\infty \rangle}(\psi_h) \\ &= \lim_{h \rightarrow 0} J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle}(\psi_h) = \Gamma \left( (J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle})_{h>0} \right) (\psi) < +\infty. \end{aligned} \quad (5.4.135)$$

Now we have proved both implications of the claim:  $\psi$  is admissible for the functional  $\Gamma \left( (J_{\mathbf{S}, h}^{2d, \langle 2, +\infty \rangle})_{h>0} \right)$  iff there exists  $\mathbf{S}_0$  such that  $(\psi, \mathbf{S}_0) \in \mathcal{B}_0^{2d, \langle 2, +\infty \rangle}$ . Clearly

this is equivalent to the definition of the set  $\mathcal{B}_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle}$ . From inequalities (5.4.134) and (5.4.135) we obtain that for any  $\psi \in \mathcal{B}_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle}$  and its appropriate  $\mathbf{S}_0$  (such that  $(\psi, \mathbf{S}) \in \mathcal{B}_0^{2d,\langle 2,+\infty \rangle}$ ) we have

$$\Gamma \left( (J_{\mathbf{S},h}^{2d,\langle 2,+\infty \rangle})_{h>0} \right) (\psi) = J_0^{2d,\langle 2,+\infty \rangle}(\psi, \mathbf{S}_0) = - \int_{\omega} \mathbf{f} \cdot \psi \sqrt{a} d\mathbf{x}' = J_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle}(\psi), \quad (5.4.136)$$

which completes the proof.  $\square$

## 5.5. TECHNICAL LEMMAS FOR MATRICES

**Lemma 5.5.1.** Let there be matrix functions  $\mathbf{A}, \mathbf{A}_h = \mathbf{A} + h\mathbf{X} \in \mathbb{M}_{3 \times 3}$ , and let  $\mathbf{A}$  be invertible on its domain. Then

$$\|\mathbf{A}^{-1} - \mathbf{A}_h^{-1}\|_F \leq Ch(1 - Ch\|\mathbf{X}\|_F\|\mathbf{A}^{-1}\|_F)^{-1}\|\mathbf{X}\|_F\|\mathbf{A}^{-1}\|_F^2, \quad (5.5.1)$$

where  $C$  is independent of  $h, \mathbf{A}$  and  $\mathbf{X}$ .

*Proof.* If  $\mathbf{A}$  is invertible, then for sufficiently small  $h$  the matrix  $\mathbf{A}_h$  is invertible as well since  $GL(3)$  is open in the set of matrices. Using submultiplicativity of the 2-norm, equivalence of the matrix norms, and identity  $\mathbf{A}_h^{-1} - \mathbf{A}^{-1} = -h\mathbf{A}_h^{-1}\mathbf{X}\mathbf{A}^{-1}$ , we obtain

$$\|\mathbf{A}_h^{-1}\|_F - \|\mathbf{A}^{-1}\|_F \leq \|\mathbf{A}^{-1} - \mathbf{A}_h^{-1}\|_F \leq Ch\|\mathbf{X}\|_F\|\mathbf{A}^{-1}\|_F\|\mathbf{A}_h^{-1}\|_F,$$

for  $C > 1$  coming from the norm equivalence. From here we obtain bound

$$\|\mathbf{A}_h^{-1}\|_F \leq \|\mathbf{A}^{-1}\|_F(1 - Ch\|\mathbf{X}\|_F\|\mathbf{A}^{-1}\|_F)^{-1}, \quad (5.5.2)$$

which we plug in again in the same inequality to obtain bound on  $\|\mathbf{A}^{-1} - \mathbf{A}_h^{-1}\|_F$ .  $\square$

**Lemma 5.5.2.** Let  $\mathbf{A}, \mathbf{B}, \mathbf{A}_h = \mathbf{A} + h\mathbf{X}, \mathbf{B}_h = \mathbf{B} + h\mathbf{Y}$  be matrix functions, such that  $\mathbf{A}, \mathbf{X} \in L^2(\omega; \text{SO}(3)), \mathbf{B}, \mathbf{Y} \in L^\infty(\omega; \text{SO}(3)), \mathbf{B}$  invertible a.e. Then

$$\begin{aligned} & \|\mathbf{A}_h\mathbf{B}_h^{-1} - \mathbf{A}\mathbf{B}^{-1}\|_{L^2(\omega; \mathbb{R}^{3 \times 3})}^2 \leq \\ & C \left( \|\mathbf{A}\|_{L^2(\omega; \mathbb{R}^{3 \times 3})}^2 \|\mathbf{B}_h - \mathbf{B}\|_{L^\infty(\omega; \mathbb{R}^{3 \times 3})}^2 \|\mathbf{B}^{-1}\|_{L^\infty(\omega; \mathbb{R}^{3 \times 3})}^4 \right. \\ & \quad \left. + \|\mathbf{A}_h - \mathbf{A}\|_{L^2(\omega; \mathbb{R}^{3 \times 3})}^2 \|\mathbf{B}^{-1}\|_{L^\infty(\omega; \mathbb{R}^{3 \times 3})}^2 \right). \end{aligned} \quad (5.5.3)$$

*Proof.* For  $h$  small enough we have

$$\begin{aligned} & \|\mathbf{A}_h\mathbf{B}_h^{-1} - \mathbf{A}\mathbf{B}^{-1}\|_{L^2(\omega; \mathbb{R}^{3 \times 3})}^2 \leq \\ & C \left( \|\mathbf{A}\|_{L^2(\omega; \mathbb{R}^{3 \times 3})}^2 \|\mathbf{B}_h^{-1} - \mathbf{B}^{-1}\|_{L^\infty(\omega; \mathbb{R}^{3 \times 3})}^2 + \|\mathbf{B}_h^{-1}\|_{L^\infty(\omega; \mathbb{R}^{3 \times 3})}^2 \|\mathbf{A}_h - \mathbf{A}\|_{L^2(\omega; \mathbb{R}^{3 \times 3})}^2 \right). \end{aligned}$$

For  $\|\mathbf{B}_h^{-1}\|$  we use the estimate (5.5.2) and for the difference of matrix inverses we use Lemma 5.5.1.  $\square$

In the sequel we give a series of lemmas for vectors  $P_1, P_2 \in \mathbb{R}^3$ , matrix  $\mathbf{Q} \in \text{SO}(3)$  and regular matrix  $\mathbf{M} \in \mathbb{M}^{3 \times 3}$  such that  $M_3 = \frac{M_1 \times M_2}{\|M_1 \times M_2\|}$  (with notation  $X_i := \mathbf{X}e_i$ ). If matrix  $\begin{bmatrix} P_1 & P_2 \end{bmatrix}$  has full rank, we introduce vector  $P_3 := \frac{P_1 \times P_2}{\|P_1 \times P_2\|}$ . We will also use the notation

$$\mathbf{T} := \begin{bmatrix} P_1 & P_2 & \mathbf{Q}M_3 \end{bmatrix} \quad \text{and} \quad \mathbf{N} = \mathbf{Q}\mathbf{M}. \quad (5.5.4)$$

These matrices will play role for (a.e.  $\mathbf{x}' \in \omega$ ):  $P_1 = \partial_1 \psi(\mathbf{x}')$ ,  $P_2 = \partial_2 \psi(\mathbf{x}')$ ,  $P_3 = a_3^{\psi(\mathbf{x}')}(\mathbf{x}')$  (if exists),  $\mathbf{Q} = \mathbf{S}(\mathbf{x}')$ ,  $\mathbf{M} = \begin{bmatrix} a_1(\mathbf{x}') & a_2(\mathbf{x}') & a_3(\mathbf{x}') \end{bmatrix}$ .

**Lemma 5.5.3.** Let there be regular  $\mathbf{P} = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} \in \mathbb{M}^{3 \times 3}$ ,  $\mathbf{Q} \in \text{SO}(3)$  and regular  $\mathbf{M} \in \mathbb{M}^{3 \times 3}$  with  $P_3 = \frac{P_1 \times P_2}{\|P_1 \times P_2\|}$ ,  $M_3 = \frac{M_1 \times M_2}{\|M_1 \times M_2\|}$ . Then we have following equivalences:

$$\mathbf{Q}M_3 \cdot P_\beta = 0, \beta = 1, 2 \quad \Longleftrightarrow \quad \mathbf{Q}M_\beta \cdot P_3 = 0, \beta = 1, 2 \quad \Longleftrightarrow \quad \mathbf{Q}M_3 = P_3. \quad (5.5.5)$$

*Proof.* The last condition implies the first two since  $P_3 \perp P_\beta$ ,  $M_3 \perp M_\beta$  for  $\beta = 1, 2$  from the definition of  $\mathcal{M}$ . Again since  $P_3 \perp P_\beta$ ,  $M_3 \perp M_\beta$ , the first two conditions (independently) imply  $P_3 \parallel \mathbf{Q}M_3$ . Since  $\det \begin{bmatrix} P_1 & P_2 & \mathbf{Q}M_3 \end{bmatrix} \geq 0$  and vectors  $P_3$  and  $\mathbf{Q}M_3$  are unit vectors, we must have  $P_3 = \mathbf{Q}M_3$ .  $\square$

**Lemma 5.5.4.** Let there be  $P_1, P_2 \in \mathbb{R}^3$  and  $\mathbf{M} \in \mathbb{M}^{3 \times 3}$  regular with  $M_3 = \frac{M_1 \times M_2}{\|M_1 \times M_2\|}$ . Then there exists  $\mathbf{Q} \in \text{SO}(3)$  such that

$$P_1 \cdot \mathbf{Q}M_2 = P_2 \cdot \mathbf{Q}M_1, \quad P_\beta \cdot \mathbf{Q}M_3 = 0, \beta = 1, 2.$$

*Proof.* If  $\overline{\mathbf{P}} := \begin{bmatrix} P_1 & P_2 \end{bmatrix}$  is zero matrix, every  $\mathbf{Q}$  satisfies the condition. Let matrices  $\overline{\mathbf{Q}}_P \in \mathbb{M}^{3 \times 2}$  and  $\overline{\mathbf{R}}_P \in \mathbb{M}^{2 \times 2}$  be from QR-decompositions of matrix  $\overline{\mathbf{P}}$ . Additionally, let diagonal elements of  $\overline{\mathbf{R}}_P$  be nonnegative. Let us define

$$\mathbf{Q}_P = \begin{bmatrix} \overline{\mathbf{Q}}_P & \mathbf{v} \end{bmatrix}, \quad \mathbf{R}_P = \begin{bmatrix} \overline{\mathbf{R}}_P & 0 \\ 0 & 1 \end{bmatrix}, \quad (5.5.6)$$

where  $\mathbf{v}$  is such that  $\mathbf{Q}_P \in \text{SO}(3)$ . Now we define  $\mathbf{P} := \mathbf{Q}_P \mathbf{R}_P$ . We can see that  $P_3 \perp P_\beta$ ,  $\|P_3\| = 1$  and that  $\mathbf{P}$  has nonnegative determinant. Specially, if matrix  $\overline{\mathbf{P}}$  is full rank, then the last column of the matrix  $\mathbf{P}$  is

$$P_3 = \frac{P_1 \times P_2}{\|P_1 \times P_2\|}. \quad (5.5.7)$$

Let us define  $\mathbf{Q}_M \in \text{SO}(3)$ ,  $\mathbf{R}_M \in \mathbb{R}^{3 \times 3}$  from QR-decomposition of the matrix  $\mathbf{M}$  (with nonnegative diagonal elements in  $\mathbf{R}_M$ ). Similarly as in the case of the full-rank matrix  $\bar{\mathbf{P}}$ , matrix  $\mathbf{R}_M$  is block diagonal, with down right element equal to 1. Let  $\bar{\mathbf{Q}}_M$  denote the first two columns of  $\mathbf{Q}_M$  and  $\bar{\mathbf{R}}_M$  first  $2 \times 2$  block of the matrix  $\mathbf{R}_M$ .

The condition  $P_1 \cdot \mathbf{Q}M_2 = P_2 \cdot \mathbf{Q}M_1$  can be written equivalently as

$$\bar{\mathbf{R}}_P^T \bar{\mathbf{Q}}_P^T \mathbf{Q} \bar{\mathbf{Q}}_M \bar{\mathbf{R}}_M = \bar{\mathbf{R}}_M^T \bar{\mathbf{Q}}_M^T \mathbf{Q}^T \bar{\mathbf{Q}}_P \bar{\mathbf{R}}_P. \quad (5.5.8)$$

Let us define  $\bar{\tilde{\mathbf{Q}}} := \bar{\mathbf{Q}}_P^T \mathbf{Q} \bar{\mathbf{Q}}_M$ . We further have

$$\bar{\mathbf{R}}_P^T \bar{\tilde{\mathbf{Q}}} \bar{\mathbf{R}}_M = \bar{\mathbf{R}}_M^T \bar{\tilde{\mathbf{Q}}}^T \bar{\mathbf{R}}_P. \quad (5.5.9)$$

We take SVD-decomposition of the matrix  $\bar{\mathbf{R}}_P \bar{\mathbf{R}}_M^{-1}$ . Note that we can assume that unitary matrices in the decomposition have determinant equal to 1. We now have

$$\mathbf{V} \Sigma \mathbf{U}^T \bar{\tilde{\mathbf{Q}}} = \bar{\tilde{\mathbf{Q}}}^T \mathbf{U} \Sigma \mathbf{V}^T, \quad (5.5.10)$$

which implies

$$\Sigma \mathbf{R} = \mathbf{R}^T \Sigma. \quad (5.5.11)$$

for  $\mathbf{R} = \mathbf{U}^T \bar{\tilde{\mathbf{Q}}} \mathbf{V}$ . By using that all unitary  $2 \times 2$  matrices with positive determinant can be written in the form

$$\begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}$$

for  $\varphi \in [0, 2\pi]$ , the equation  $\Sigma \mathbf{R} = \mathbf{R}^T \Sigma$  componentwise implies that necessarily we have  $\text{tr}(\Sigma) \sin \varphi = 0$ . Since  $\mathbf{P} \neq 0$ , as consequence  $\Sigma \neq 0$ , so we necessarily have  $\sin \varphi = 0 \implies \cos \varphi = \pm 1$ . Thus it is necessarily  $\mathbf{R} = \mathbf{I}$  or  $\mathbf{R} = -\mathbf{I}$ .

From that we have  $\bar{\tilde{\mathbf{Q}}} = \pm \mathbf{U} \mathbf{V}^T$ . Let us define

$$\tilde{\mathbf{Q}}_{\pm} = \begin{bmatrix} \pm \mathbf{U} \mathbf{V}^T & 0 \\ 0 & 1 \end{bmatrix}, \quad (5.5.12)$$

and  $\mathbf{Q}_{\pm} := \mathbf{Q}_P \tilde{\mathbf{Q}}_{\pm} \mathbf{Q}_M$ . We claim that both those matrices satisfy conditions of the lemma. For first condition we see by direct calculations as above. For the third equation, we observe the (3, 3) element in the equation

$$\mathbf{P}^T \mathbf{Q}_{\pm} \mathbf{M} = \mathbf{R}_P^T \mathbf{Q}_P^T \mathbf{Q}_{\pm} \mathbf{Q}_M \mathbf{R}_M = \mathbf{R}_P^T \tilde{\mathbf{Q}}_{\pm} \mathbf{R}_M, \quad (5.5.13)$$

and conclude that  $P_3 \cdot \mathbf{Q}_\pm M_3 = 1$ , which implies  $P_3 = \mathbf{Q}_\pm M_3$  (since both vectors are unit vectors). Since  $P_3 \perp P_\beta$ , the proof is done.  $\square$

**Remark 5.5.5.** From the proof we see that there are actually at least two choices for such  $\mathbf{Q}$ . If  $\bar{\mathbf{P}}$  is regular, one can see that there are only two choices  $\mathbf{Q}_\pm$ , and that they satisfy  $\mathbf{Q}_+ M_\beta = -\mathbf{Q}_- M_\beta$ ,  $\beta = 1, 2$ ,  $\mathbf{Q}_+ M_3 = \mathbf{Q}_- M_3$ .

**Lemma 5.5.6.** Let  $P_1, P_2, \mathbf{Q}$  and  $\mathbf{M}$  satisfy the statement of Lemma 5.5.4. Additionally, let  $\mathbf{T}\mathbf{M}^{-1} \in \text{SO}(3)$ , where  $\mathbf{T} = \begin{bmatrix} P_1 & P_2 & \mathbf{Q}M_3 \end{bmatrix}$ . Then we have

$$\mathbf{P} = \mathbf{Q}\mathbf{M} \quad \text{or} \quad \mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{Q}\mathbf{M}. \quad (5.5.14)$$

*Proof.* Since  $\mathbf{T}\mathbf{M}^{-1} \in \text{SO}(3)$  the matrix  $\mathbf{T}$  is regular and thus  $P_3 = \frac{P_1 \times P_2}{\|P_1 \times P_2\|}$  is well defined. Then note that from Lemma 5.5.3 we have  $P_3 = \mathbf{Q}M_3$ , which implies that  $\mathbf{T} = \mathbf{P}$ . Thus we have that  $\mathbf{P}\mathbf{M}^{-1} \in \text{SO}(3)$ . Then  $\mathbf{U} := \mathbf{P}\mathbf{M}^{-1}\mathbf{Q}^{-1}$  is obviously an element of  $\text{SO}(3)$ . We also have

$$\mathbf{U}\mathbf{Q}M_3 = \mathbf{P}\mathbf{M}^{-1}\mathbf{Q}^{-1}\mathbf{Q}M_3 = \mathbf{P}\mathbf{M}^{-1}M_3 = \mathbf{P}e_3 = P_3 = \mathbf{Q}M_3.$$

Also note that since  $\mathbf{P}\mathbf{M}^{-1} \in \text{SO}(3)$  we have  $\mathbf{P}^T\mathbf{P} = \mathbf{M}^T\mathbf{M}$ . Furthermore, by the assumptions on  $P_i$  and  $M_j$ , we have

$$\mathbf{U}\mathbf{Q}M_1 \cdot \mathbf{Q}M_2 = P_1 \cdot \mathbf{Q}M_2 = P_2 \cdot \mathbf{Q}M_1 = \mathbf{U}\mathbf{Q}M_2 \cdot \mathbf{Q}M_1,$$

$$\mathbf{U}\mathbf{Q}M_\beta \cdot \mathbf{Q}M_3 = P_\beta \cdot \mathbf{Q}M_3 = P_\beta \cdot P_3 = 0,$$

$$\mathbf{U}\mathbf{Q}M_3 \cdot \mathbf{Q}M_\beta = P_3 \cdot \mathbf{Q}M_\beta = \mathbf{Q}M_3 \cdot \mathbf{Q}M_\beta = M_3 \cdot M_\beta = 0.$$

Thus we conclude that assumptions imply  $\mathbf{U} \in \text{SO}(3)$ ,  $\mathbf{U}^T = \mathbf{U}$ ,  $\mathbf{U}(\mathbf{Q}M_3) = (\mathbf{Q}M_3)$ .

First two conditions imply  $\mathbf{U}^2 = \mathbf{I}$ , so all eigenvalues of  $\mathbf{U}$  are  $\pm 1$ . Since  $\mathbf{Q}M_3$  is an eigenvector of  $\mathbf{U}$  for eigenvalue 1 and since  $\det \mathbf{U} = 1$ , the only possibilities are that both other eigenvalues (for eigenspace  $\text{span}\{\mathbf{Q}M_1, \mathbf{Q}M_2\}$ ) are either 1 or  $-1$ , leading to the only solutions as in (5.5.14).  $\square$

**Lemma 5.5.7.** Let there be vectors  $P_1, P_2 \in \mathbb{R}^3$  and matrices  $\mathbf{Q} \in \text{SO}(3)$  and regular  $\mathbf{M} \in \mathbb{M}^{3 \times 3}$  with  $M_3 = \frac{M_1 \times M_2}{\|M_1 \times M_2\|}$ . Additionally let  $\mathbf{T}$  and  $\mathbf{N}$  be defined as in (5.5.4). Let



$\det \mathbf{T} > 0$ . For a general matrix  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$  let  $\bar{\mathbf{X}} \in \mathbb{M}^{3 \times 2}$  denotes matrix made up of the first two columns. Then there are positive constants  $c, C$  independent of  $P_1, P_2, \mathbf{Q}$  and  $\mathbf{M}$  such that

$$\|\bar{\mathbf{T}}^T \bar{\mathbf{T}} - \bar{\mathbf{N}}^T \bar{\mathbf{N}}\|_F^2 + \|\bar{\mathbf{T}}^T \mathbf{Q} M_3\|_F^2 \geq c \|\mathbf{N}^{-1}\|_F^{-4} \text{dist}^2(\mathbf{T} \mathbf{N}^{-1}, \text{SO}(3)) \quad (5.5.15)$$

and

$$\|\bar{\mathbf{T}}^T \bar{\mathbf{T}} - \bar{\mathbf{N}}^T \bar{\mathbf{N}}\|_F^2 + \|\bar{\mathbf{T}}^T \mathbf{Q} M_3\|_F^2 \leq C(\|\mathbf{T}\|_F^2 \|\mathbf{N}^{-1}\|_F^2 + 1) \|\mathbf{N}\|_F^4 \text{dist}^2(\mathbf{T} \mathbf{N}^{-1}, \text{SO}(3)). \quad (5.5.16)$$

*Proof.* From definitions of  $\mathbf{T}$  and  $\mathbf{N}$  we have

$$\mathbf{T}^T \mathbf{T} - \mathbf{N}^T \mathbf{N} = \begin{bmatrix} \bar{\mathbf{T}}^T \bar{\mathbf{T}} & \bar{\mathbf{T}}^T \mathbf{Q} M_3 \\ (\mathbf{Q} M_3)^T \bar{\mathbf{T}} & 1 \end{bmatrix} - \begin{bmatrix} \bar{\mathbf{N}}^T \bar{\mathbf{N}} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{T}}^T \bar{\mathbf{T}} - \bar{\mathbf{N}}^T \bar{\mathbf{N}} & \bar{\mathbf{T}}^T \mathbf{Q} M_3 \\ (\mathbf{Q} M_3)^T \bar{\mathbf{T}} & 0 \end{bmatrix}$$

Thus it is sufficient to prove

$$C \|\mathbf{N}\|_F^4 \text{dist}^2(\mathbf{T} \mathbf{N}^{-1}, \text{SO}(3)) \geq \|\mathbf{T}^T \mathbf{T} - \mathbf{N}^T \mathbf{N}\|_F^2 \geq c \|\mathbf{N}^{-1}\|_F^{-4} \text{dist}^2(\mathbf{T} \mathbf{N}^{-1}, \text{SO}(3)).$$

The inequality  $\|\mathbf{AB}\|_2 \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2$  holds for any general matrices, so due to equivalence of the norms we have that  $\|\mathbf{AB}\|_F \leq c_1 \|\mathbf{A}\|_F \|\mathbf{B}\|_F$ . Thus from Lemma 5.5.9 we have

$$\begin{aligned} \|\mathbf{T}^T \mathbf{T} - \mathbf{N}^T \mathbf{N}\|_F^2 &= \|\mathbf{N}^T (\mathbf{N}^{-T} \mathbf{T}^T \mathbf{T} \mathbf{N}^{-1} - \mathbf{I}_3) \mathbf{N}\|_F^2 \\ &\geq c \|\mathbf{N}^{-1}\|_F^{-4} \|\mathbf{N}^{-T} \mathbf{T}^T \mathbf{T} \mathbf{N}^{-1} - \mathbf{I}_3\|_F^2 \\ &\geq c' \|\mathbf{N}^{-1}\|_F^{-4} \text{dist}^2(\mathbf{T} \mathbf{N}^{-1}, \text{SO}(3)). \end{aligned} \quad (5.5.17)$$

The other inequality is obtained similarly, again using equality  $\|\mathbf{T}^T \mathbf{T} - \mathbf{N}^T \mathbf{N}\|_F^2 = \|\mathbf{N}^T (\mathbf{N}^{-T} \mathbf{T}^T \mathbf{T} \mathbf{N}^{-1} - \mathbf{I}_3) \mathbf{N}\|_F^2$ , using the submultiplicativity

$$\begin{aligned} \|\mathbf{T}^T \mathbf{T} - \mathbf{N}^T \mathbf{N}\|_F^2 &= \|\mathbf{N}^T (\mathbf{N}^{-T} \mathbf{T}^T \mathbf{T} \mathbf{N}^{-1} - \mathbf{I}_3) \mathbf{N}\|_F^2 \\ &\leq c (\|\mathbf{N}^T\|_F^4 \|\mathbf{N}^{-T} \mathbf{T}^T \mathbf{T} \mathbf{N}^{-1} - \mathbf{I}_3\|_F^2) \end{aligned}$$

and the other bound from Lemma 5.5.9.  $\square$

**Remark 5.5.8.** In our case,  $\mathbf{N} = \mathbf{S}(\mathbf{x}') \begin{bmatrix} \mathbf{a}_1(\mathbf{x}') & \mathbf{a}_2(\mathbf{x}') & \mathbf{a}_3(\mathbf{x}') \end{bmatrix}$ , so we conclude  $\|\mathbf{N}^{-1}\|_2 = \max\{1, \sqrt{\rho(\mathbf{A}^c)}\}$  and  $\|\mathbf{N}\|_2 = \max\{1, \sqrt{\rho(\mathbf{A}_c)}\}$ , where  $\rho(\mathbf{X})$  denotes the spectral radius of the matrix  $\mathbf{X}$  (the largest absolute eigenvalue). In other words, the constant is bounded only by the properties of the undeformed geometry.

**Lemma 5.5.9.** Let  $m, n \in \mathbb{N}$ ,  $m \leq n \leq 3$ . Let  $O(m, n)$  denote the set of matrices in  $\mathbb{M}^{n \times m}$  with orthonormal columns. Then for all  $\mathbf{X} \in \mathbb{M}^{n \times m}$  bounds

$$C(\|\mathbf{X}\|_F^2 + 1) \text{dist}^2(\mathbf{X}, O(m, n)) \geq \|\mathbf{X}^T \mathbf{X} - \mathbf{I}\|_F^2 \geq \text{dist}^2(\mathbf{X}, O(m, n)) \quad (5.5.18)$$

hold with a positive constant  $C$  independent of  $\mathbf{X}$ . If  $m = n = 3$  and  $\det \mathbf{X} \geq 0$ , the set  $O(m, n)$  can be substituted by  $\text{SO}(3)$ .

*Proof.* Let  $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$  be a SVD decomposition of  $\mathbf{X}$ , and  $\sigma_i$   $i = 1, \dots, m$  its singular values. If  $\mathbf{X}$  is quadratic and its determinant is nonnegative, then  $\mathbf{U}$  and  $\mathbf{V}$  can be chosen to have positive determinant. From the properties of Frobenius norm we have

$$\text{dist}^2(\mathbf{X}, \text{SO}(3)) = \text{dist}^2(\mathbf{\Sigma}, \text{SO}(3)) = \|\mathbf{\Sigma} - \mathbf{I}\|_F^2 = \sum_{i=1}^m (\sigma_i - 1)^2, \quad (5.5.19)$$

and

$$\begin{aligned} \|\mathbf{X}^T \mathbf{X} - \mathbf{I}\|_F^2 &= \|\mathbf{V}^T \mathbf{\Sigma}^T \mathbf{\Sigma} \mathbf{V} - \mathbf{I}\|_F^2 = \|\mathbf{\Sigma}^T \mathbf{\Sigma} - \mathbf{I}\|_F^2 \\ &= \sum_{i=1}^m (\sigma_i^2 - 1)^2 = \sum_{i=1}^m (\sigma_i - 1)^2 (\sigma_i + 1)^2. \end{aligned} \quad (5.5.20)$$

Since for all  $i = 1, \dots, m$  we have  $0 \leq \sigma_i \leq \sigma_{\max} = \|\mathbf{X}\|_2$ , and since all matrix norms are equivalent, we have

$$\sum_{i=1}^m (\sigma_i - 1)^2 \leq \sum_{i=1}^m (\sigma_i^2 - 1)^2 \leq (\sigma_{\max} + 1)^2 \sum_{i=1}^m (\sigma_i - 1)^2 \leq C(\|\mathbf{X}\|_F^2 + 1) \sum_{i=1}^m (\sigma_i - 1)^2. \quad (5.5.21)$$

Together with (5.5.19) and (5.5.20), we obtain the desired bounds. In the case of  $\mathbf{X}$  with nonnegative determinant, since  $\mathbf{U}$  and  $\mathbf{V}$  have positive determinant, all calculations stay the same.  $\square$

## 6. NONLINEAR 3D–2D MODEL

### 6.1. PROBLEM SETUP

After defining a Naghdi type nonlinear shell model in Chapter 5, our goal in this chapter is to define a nonlinear 3d–2d interaction model for which, as we did in the Part I of the thesis, we will prove the same asymptotic properties for the structure made of the 3d body and a thin 3d body described in the Chapter 4. and proved in the Theorem 4.2.1.

As before, let us define  $\omega = \langle 0, 1 \rangle^2$ ,  $\Omega^0 = \omega \times \langle -1, 0 \rangle$ , and  $\Gamma^0 = \omega \times \{0\}$ . Let us take  $p \in \langle 1, 4 \rangle$ . For arbitrary  $h > 0$  and  $\alpha > -1$  we define a functional  $K_h^{2d, \alpha} : L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3)) \rightarrow \overline{\mathbb{R}}$  as a sum of functionals  $\mathcal{W}^0$  defined in (4.1.3) and  $J_h^{2d, \alpha}$  defined in (5.4.1) with appropriate set of admissible functions. More precisely,

$$K_h^{2d, \alpha}(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) = \begin{cases} \int_{\Omega^0} W^0(\nabla \psi) + h^{-\alpha} \int_{\omega} w_m(\tilde{\mathbf{S}}, \nabla' \tilde{\psi}) d\mathbf{x}' \\ \quad + h^{-\alpha+2} \frac{1}{12} \int_{\omega} q_f(\tilde{\mathbf{S}}, \nabla' \tilde{\mathbf{S}}) d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \tilde{\psi} d\mathbf{x}' & (\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \in \mathcal{A}^{2d}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.1.1)$$

where

$$\begin{aligned} \mathcal{A}^{2d} := \{(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{1,4}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3)) : \\ \psi|_{x_3=0} = \tilde{\psi}, \psi|_{x_3=-1} = \text{id}, \det \begin{bmatrix} \tilde{\mathbf{S}} e_3 & \partial_1 \tilde{\psi} & \partial_2 \tilde{\psi} \end{bmatrix} \geq 0\}. \end{aligned} \quad (6.1.2)$$

The parameter  $\alpha > -1$  again serves to measure the order of magnitude of Lamé coefficients for the thin part of the structure, as it was explained in the definition of functionals  $J_h^{2d, \alpha}$  in (5.4.1).

The functions  $w_m$  and  $q_f$  are the ones defined in Chapter 5 for the simple parametrization of the middle surface of the undeformed geometry of the shell:  $\boldsymbol{\theta}(\mathbf{x}') = (\mathbf{x}', 0)$ . More precisely:

$$w_m(\tilde{\mathbf{S}}, \nabla' \tilde{\boldsymbol{\psi}}) = W_m \left( \begin{bmatrix} (\tilde{\mathbf{S}}^T \nabla' \tilde{\boldsymbol{\psi}} + \mathbf{I}_{3,2})^T (\tilde{\mathbf{S}}^T \nabla' \tilde{\boldsymbol{\psi}} - \mathbf{I}_{3,2}) \\ (\tilde{\mathbf{S}} \mathbf{e}_3)^T \nabla' \tilde{\boldsymbol{\psi}} \end{bmatrix} \right),$$

$$q_f(\tilde{\mathbf{S}}, \nabla' \tilde{\mathbf{S}}) = Q_f(\tilde{\mathbf{S}}^T \nabla' \tilde{\mathbf{S}}) \quad (6.1.3)$$

for some continuous functions  $Q_f : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$  and  $W_m : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$  satisfying conditions N1), N2) and N3). For  $W^0$  we assume W0.2), W0.3) and W0.4), and for the forcing term we assume  $\mathbf{f} \in L^{4/3}(\omega; \mathbb{R}^3)$ . We are interested in the  $\Gamma$ -limit of the family of functionals  $(K_h^{2d,\alpha})_{h>0}$  as  $h \rightarrow 0$ , for various cases of  $\alpha$ .

To make results more convenient to compare with results of the Theorem 4.2.1, and since some results will not be proven in this form, we will also observe family of functionals  $(K_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  for which  $K_{\mathbf{S},h}^{2d,\alpha}(\tilde{\boldsymbol{\psi}}) = \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} K_h^{2d,\alpha}(\tilde{\boldsymbol{\psi}}, \tilde{\mathbf{S}})$ . More precisely, for  $h > 0$  and  $\alpha > -1$  we define  $K_{\mathbf{S},h}^{2d,\alpha} : L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  by

$$K_{\mathbf{S},h}^{2d,\alpha}(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) = \begin{cases} \int_{\Omega^0} W^0(\nabla \boldsymbol{\psi}) + \inf_{\mathbf{S} \in L^2(\omega; \text{SO}(3))} \left[ h^{-\alpha} \int_{\omega} w_m(\tilde{\mathbf{S}}, \nabla' \tilde{\boldsymbol{\psi}}) d\mathbf{x}' \right. \\ \quad \left. + h^{-\alpha+2} \frac{1}{12} \int_{\omega} q_f(\tilde{\mathbf{S}}, \nabla' \tilde{\mathbf{S}}) d\mathbf{x}' \right] - \int_{\omega} \mathbf{f} \cdot \tilde{\boldsymbol{\psi}} d\mathbf{x}' & (\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) \in \mathcal{A}_{\mathbf{S}}^{2d}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.1.4)$$

where

$$\mathcal{A}_{\mathbf{S}}^{2d} := \{(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{1,4}(\omega; \mathbb{R}^3) : \\ \exists \tilde{\mathbf{S}} \in W^{1,2}(\omega; \text{SO}(3)) \text{ s.t. } (\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}, \tilde{\mathbf{S}}) \in \mathcal{A}^{2d}\}. \quad (6.1.5)$$

We will also determine the  $\Gamma$ -limit of the family of functionals  $(K_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  as  $h \rightarrow 0$ , for various cases of  $\alpha$ .

## 6.2. THE MAIN RESULT

In this section we are going to determine the  $\Gamma$ –limit of families of  $(K_h^{2d,\alpha})_{h>0}$  and  $(K_{S,h}^{2d,\alpha})_{h>0}$  for different values of  $\alpha > -1$ .

**Theorem 6.2.1.** Let the family of functionals  $(K_h^{2d,\alpha})_{h>0}$  be as defined in (6.1.1). Let us assume that conditions N1), N2), N3), W0.2), W0.3) and W0.4) are satisfied.

a) Let  $\alpha \in \langle -1, 0 \rangle$ . Then the  $\Gamma$ –limit of the family of functionals  $(K_h^{2d,\langle -1,0 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology is

$$K_0^{2d,\langle -1,0 \rangle}(\psi, \tilde{\psi}, \tilde{S}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\psi} d\mathbf{x}' & (\psi, \tilde{\psi}, \tilde{S}) \in \mathcal{A}_0^{2d,\langle -1,0 \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.2.1)$$

where

$$\mathcal{A}_0^{2d,\langle -1,0 \rangle} := \{(\psi, \tilde{\psi}, \tilde{S}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3)) : \psi|_{x_3=-1} = \text{id}, \psi|_{x_3=0} = \tilde{\psi}\}. \quad (6.2.2)$$

b) Let  $\alpha = 0$ . Then the  $\Gamma$ –limit of the family of functionals  $(K_h^{2d,0})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology is

$$K_0^{2d,0}(\psi, \tilde{\psi}, \tilde{S}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} + \Gamma - \left( \int_{\omega} w_m(\tilde{S}, \nabla' \tilde{\psi}) d\mathbf{x}' \right) - \int_{\omega} \mathbf{f} \cdot \tilde{\psi} d\mathbf{x}' & (\psi, \tilde{\psi}, \tilde{S}) \in \mathcal{A}_0^{2d,0}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.2.3)$$

where

$$\mathcal{A}_0^{2d,0} = \{(\psi, \tilde{\psi}, \tilde{S}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{1,4}(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3)) : \psi|_{x_3=0} = \tilde{\psi}, \psi|_{x_3=-1} = \text{id}, \det \begin{bmatrix} \tilde{S}e_3 & \partial_1 \tilde{\psi} & \partial_2 \tilde{\psi} \end{bmatrix} \geq 0\}. \quad (6.2.4)$$

c) Let  $\alpha = 2$ . Then the  $\Gamma$ –limit of the family of functionals  $(K_h^{2d,2})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology is

$$K_0^{2d,2}(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} + \frac{1}{12} \int_{\omega} q_f(\tilde{\mathbf{S}}, \nabla' \tilde{\mathbf{S}}) d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \tilde{\psi} d\mathbf{x}' & (\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \in \mathcal{A}_0^{2d,2}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.2.5)$$

where

$$\begin{aligned} \mathcal{A}_0^{2d,2} &:= \{(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{2,2}(\omega; \mathbb{R}^3) \times W^{1,2}(\omega; \text{SO}(3)) : \\ &\psi|_{x_3=-1} = \text{id}, \psi|_{x_3=0} = \tilde{\psi}, \\ &\mathbf{S}\mathbf{a}_\beta = \partial_\beta \psi, \beta = 1, 2 \text{ or } \mathbf{S}\mathbf{a}_\beta = -\partial_\beta \psi, \beta = 1, 2\}. \end{aligned} \quad (6.2.6)$$

d) Let  $\alpha \in \langle 2, +\infty \rangle$ . Then the  $\Gamma$ –limit of the family of functionals  $(K_h^{2d,\langle 2, +\infty \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  topology is

$$K_0^{2d,\langle 2, +\infty \rangle}(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\psi} d\mathbf{x}' & (\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \in \mathcal{A}_0^{2d,\langle 2, +\infty \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.2.7)$$

where

$$\begin{aligned} \mathcal{A}_0^{2d,\langle 2, +\infty \rangle} &:= \{(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times C(\omega; \mathbb{R}^3) \times \text{SO}(3) : \\ &\psi|_{x_3=-1} = \text{id}, \psi|_{x_3=0} = \tilde{\psi}, \tilde{\psi}(\mathbf{x}') = \tilde{\mathbf{S}} \begin{bmatrix} \mathbf{x}' & 0 \end{bmatrix}^T + \mathbf{d}, \mathbf{d} \in \mathbb{R}^3 \\ &\text{or } \tilde{\psi}(\mathbf{x}') = -\tilde{\mathbf{S}} \begin{bmatrix} \mathbf{x}' & 0 \end{bmatrix}^T + \mathbf{d}, \mathbf{d} \in \mathbb{R}^3\}. \end{aligned} \quad (6.2.8)$$

Note that we lack a result for  $\alpha \in \langle 0, 5/3 \rangle$ . This is due to the fact we lack an analogous result in the case of the  $\Gamma$ –limit of family  $(J_h^{2d,\alpha})_{h>0}$ . However, we do have determined the  $\Gamma$ –limit of family  $(J_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  in the same regime  $\alpha \in \langle 0, 5/3 \rangle$ , and all other regimes as well, so in the next theorem we will again have a result in all five regimes. Also, results in parts b), c) and d) of the previous theorem are not so easy to read due to definitions of functionals or the set of admissible functions. For this reasons we give the following theorem.

**Theorem 6.2.2.** Let the family of functionals  $(K_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  be as defined in (6.1.4). Let us assume that conditions **N1**), **N2**), **N3**), **W0.2**), **W0.3**) and **W0.4**) are satisfied.

- a) Let  $\alpha \in \langle -1, 0 \rangle$ . Then the  $\Gamma$ –limit of the family of functionals  $(K_{\mathbf{S},h}^{2d,\langle -1,0 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3)$  topology is

$$K_{\mathbf{S},0}^{2d,\langle -1,0 \rangle}(\psi, \tilde{\psi}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\psi} d\mathbf{x}' & (\psi, \tilde{\psi}) \in \mathcal{A}_{\mathbf{S},0}^{2d,\langle -1,0 \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.2.9)$$

where

$$\mathcal{A}_{\mathbf{S},0}^{2d,\langle -1,0 \rangle} := \{(\psi, \tilde{\psi}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) : \psi|_{x_3=-1} = \text{id}, \psi|_{x_3=0} = \tilde{\psi}\}. \quad (6.2.10)$$

- b) Let  $\alpha = 0$ . Then the  $\Gamma$ –limit of the family of functionals  $(K_{\mathbf{S},h}^{2d,0})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3)$  topology is

$$K_{\mathbf{S},0}^{2d,0}(\psi, \tilde{\psi}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} \\ + \int_{\omega} Qw_m^0(\nabla' \tilde{\psi}) d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \tilde{\psi} d\mathbf{x}' & (\psi, \tilde{\psi}) \in \mathcal{A}_{\mathbf{S},0}^{2d,0}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.2.11)$$

where

$$\mathcal{A}_{\mathbf{S},0}^{2d,0} = \{(\psi, \tilde{\psi}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{1,4}(\omega; \mathbb{R}^3) : \psi|_{x_3=0} = \tilde{\psi}, \psi|_{x_3=-1} = \text{id}\}, \quad (6.2.12)$$

and

$$w_m^0(\mathbf{F}) := \min_{\mathbf{S} \in \text{SO}(3)} w_m(\mathbf{S}, \mathbf{F}). \quad (6.2.13)$$

- c) Let  $\alpha \in \langle 0, 5/3 \rangle$ . Let additionally  $W_m$  satisfy  $W_m(\mathbf{Y}) \leq C\|\mathbf{Y}\|_F^2$  for some  $C > 0$ . Then the  $\Gamma$ –limit of the family of functionals  $(K_{\mathbf{S},h}^{2d,\langle 0,5/3 \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3)$  topology is

$$K_{\mathbf{S},0}^{2d,\langle 0,5/3 \rangle}(\psi, \tilde{\psi}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\psi} d\mathbf{x}' & (\psi, \tilde{\psi}) \in \mathcal{A}_{\mathbf{S},0}^{2d,\langle 0,5/3 \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.2.14)$$

where

$$\begin{aligned} \mathcal{A}_{\mathbf{S},0}^{2d,\langle 0,5/3 \rangle} &:= \{(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{1,\infty}(\omega; \mathbb{R}^3) : \\ &\boldsymbol{\psi}|_{x_3=-1} = \text{id}, \boldsymbol{\psi}|_{x_3=0} = \tilde{\boldsymbol{\psi}}, \nabla' \tilde{\boldsymbol{\psi}}^T \nabla' \tilde{\boldsymbol{\psi}} \leq \mathbf{I} \text{ a.e.} \}. \end{aligned} \quad (6.2.15)$$

d) Let  $\alpha = 2$ . Then the  $\Gamma$ –limit of the family of functionals  $(K_{\mathbf{S},h}^{2d,2})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3)$  topology is

$$K_{\mathbf{S},0}^{2d,2}(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \boldsymbol{\psi}) d\mathbf{x} \\ + \frac{1}{12} \int_{\omega} \tilde{q}_f(\mathbf{b}(\tilde{\boldsymbol{\psi}})) d\mathbf{x}' - \int_{\omega} \mathbf{f} \cdot \tilde{\boldsymbol{\psi}} d\mathbf{x}' & (\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) \in \mathcal{A}_{\mathbf{S},0}^{2d,2}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.2.16)$$

where

$$\begin{aligned} \mathcal{A}_{\mathbf{S},0}^{2d,2} &:= \{(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times W^{2,2}(\omega; \mathbb{R}^3) : \\ &\boldsymbol{\psi}|_{x_3=-1} = \text{id}, \boldsymbol{\psi}|_{x_3=0} = \tilde{\boldsymbol{\psi}}, \nabla \tilde{\boldsymbol{\psi}}^T \nabla \tilde{\boldsymbol{\psi}} = \mathbf{I}\}, \end{aligned} \quad (6.2.17)$$

$\mathbf{b}(\tilde{\boldsymbol{\psi}}) = (b_{\beta_1, \beta_2}(\tilde{\boldsymbol{\psi}}))_{\beta_1, \beta_2}$  is the curvature tensor and  $\tilde{q}_f$  is defined in (5.3.8).

e) Let  $\alpha \in \langle 2, +\infty \rangle$ . Then the  $\Gamma$ –limit of the family of functionals  $(K_{\mathbf{S},h}^{2d,\langle 2,+\infty \rangle})_{h>0}$  when  $h \rightarrow 0$  in the strong  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3)$  topology is

$$K_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle}(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \boldsymbol{\psi}) d\mathbf{x} - \int_{\omega} \mathbf{f} \cdot \tilde{\boldsymbol{\psi}} d\mathbf{x}' & (\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) \in \mathcal{A}_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle}, \\ +\infty & \text{otherwise,} \end{cases} \quad (6.2.18)$$

where

$$\begin{aligned} \mathcal{A}_{\mathbf{S},0}^{2d,\langle 2,+\infty \rangle} &:= \{(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) \in W^{1,p}(\Omega^0; \mathbb{R}^3) \times C(\omega; \mathbb{R}^3) : \\ &\boldsymbol{\psi}|_{x_3=-1} = \text{id}, \boldsymbol{\psi}|_{x_3=0} = \tilde{\boldsymbol{\psi}}, \\ &\exists \mathbf{R} \in \text{SO}(3), \mathbf{d} \in \mathbb{R}^3 \text{ s.t. } \tilde{\boldsymbol{\psi}}(\mathbf{x}') = \mathbf{R} \begin{bmatrix} \mathbf{x}' & 0 \end{bmatrix}^T + \mathbf{d}\}. \end{aligned} \quad (6.2.19)$$

Before proofs, we need one technical result.

**Corollary 6.2.3.** Let  $\tilde{\mathbf{S}} \in W^{1,2}(\omega; \text{SO}(3))$ . Then there are constants  $c > 0$  and  $C \in \mathbb{R}$  such that for all  $\tilde{\boldsymbol{\psi}} \in W^{1,4}(\omega; \mathbb{R}^3)$ ,  $h \in \langle 0, 1]$  and all  $\alpha > -1$  we have

$$\|\tilde{\boldsymbol{\psi}}\|_{L^4(\omega; \mathbb{R}^3)}^4 + J_h^{2d,\alpha}(\tilde{\boldsymbol{\psi}}, \tilde{\mathbf{S}}) \geq c \left( \|\tilde{\boldsymbol{\psi}}\|_{L^4(\omega; \mathbb{R}^3)}^4 + \chi_{\alpha \geq 0} \|\tilde{\boldsymbol{\psi}}\|_{W^{1,4}(\omega; \mathbb{R}^3)}^4 \right) - C,$$

where  $C$  is the undeformed geometry dependent.



*Proof.* For  $\alpha \geq 0$  the claim is a direct consequence of Lemma 5.4.3, nonnegativity of  $q_f$  and the fact that  $h^{-\alpha} \geq 1$  for  $h \leq 1$  and  $\alpha \geq 0$ . For  $\alpha \in \langle -1, 0 \rangle$  due to nonnegativity of  $w_m$  and  $q_f$  and Hölder inequality on the source term in (5.4.1) we have

$$\|\tilde{\psi}\|_{L^4(\omega; \mathbb{R}^3)}^4 + J_h^{2d, \alpha}(\tilde{\psi}, \tilde{\mathbf{S}}) \geq \|\tilde{\psi}\|_{L^4(\omega; \mathbb{R}^3)}^4 - M\|\tilde{\psi}\|_{L^4(\omega; \mathbb{R}^3)} \geq \frac{1}{2}\|\tilde{\psi}\|_{L^4(\omega; \mathbb{R}^3)}^4 - C,$$

by again using that  $x \mapsto ax^4 + bx$  (for  $x \geq 0$ ) is bounded from below.  $\square$

### 6.3. PROOF OF THEOREMS 6.2.1 AND 6.2.2

*Proof of Theorem 6.2.1.* We prove the theorem by using Proposition 4.3.1 for each part of the theorem separately.

Let us take any  $\alpha \in \langle -1, 0] \cup [2, +\infty)$ . Let us define family of functionals  $(J_h)_{h>0}$  by  $J_h(\phi) = J_h^{2d,\alpha}(\tilde{\psi}, \tilde{\mathbf{S}})$  (defined in (5.4.1)), for all  $(\tilde{\psi}, \tilde{\mathbf{S}}) \in \mathcal{A}^N$  (in the view of Remark 4.3.3, part 2°, each functional  $J_h$  is defined for functions  $\tilde{\psi}$  independent of the third variable and functions  $\tilde{\mathbf{S}} \in L^2(\omega; \text{SO}(3))$ ), and its set of admissible functions is  $\mathcal{A}^N$ . According to the Theorem 3.2.1, the  $\Gamma$ -limit of the family  $(J_h)_{h>0}$  is  $J_0 = J_0^{2d,\alpha}$ , depending on the value of  $\alpha$ . Due to Corollary 6.2.3, the family  $(J_h)_{h>0}$  satisfies the condition (4.3.2).

Let us define  $g(\mathbf{X}) = W^0(\mathbf{X})$ , for all  $\mathbf{X} \in \mathbb{M}^{3 \times 3}$ . Then we have  $G(\psi) = \mathcal{W}^0(\psi)$ , where  $\mathcal{W}^0$  is defined in (4.1.3), and

$$G_0(\psi) = \Gamma - \mathcal{W}^0(\psi) = \begin{cases} \int_{\Omega^0} QW^0(\nabla \psi) d\mathbf{x} & \psi \in W^{1,p}(\Omega^0; \mathbb{R}^3), \psi|_{x_3=-1} = \text{id}, \\ +\infty & \text{otherwise.} \end{cases} \quad (6.3.1)$$

Due to assumptions W0.2) and W0.4),  $g$  satisfies the nonnegativity conditions and the growth condition. Due to Lemma 4.3.6,  $G$  satisfies the condition (4.3.4).

Since for all  $\alpha$  and  $h > 0$  the functional  $K_h^{2d,\alpha}$  from (6.1.1) can be written as

$$K_h^{2d,\alpha}(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) = \begin{cases} \mathcal{W}^0(\psi) + J_h^{2d,\alpha}(\tilde{\psi}, \tilde{\mathbf{S}}) & \psi|_{x_3=0} = \tilde{\psi}, \psi|_{x_3=-1} = \text{id}, (\tilde{\psi}, \tilde{\mathbf{S}}) \in \mathcal{A}^N, \\ +\infty & \text{otherwise,} \end{cases}$$

which is of the form (4.3.5), we conclude that we have  $K_h = K_h^{2d,\alpha}$  for all  $h > 0$  and  $\mathcal{A} = \mathcal{A}^{2d,\alpha}$ . Also, functionals  $K_0^{2d,\alpha}$  from all parts of Theorem 6.2.1 are obtained as

$$K_0^{2d,\alpha}(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) = \begin{cases} \Gamma - \mathcal{W}^0(\psi) + J_0^{2d,\alpha}(\tilde{\psi}, \tilde{\mathbf{S}}) & \psi|_{x_3=0} = \tilde{\psi}|_{x_3=0}, \psi|_{x_3=-1} = \text{id}, \\ & (\tilde{\psi}, \tilde{\mathbf{S}}) \text{ admissible for } J_0^{2d,\alpha}, \\ +\infty & \text{otherwise,} \end{cases}$$

(with  $J_0^{2d,\alpha}$  is from particular parts of Theorem 5.4.1). This is of the form (4.3.7), so we conclude that  $K_0 = K_0^{2d,\alpha}$ .

Since all assumptions of Proposition 4.3.1 on  $(K_h)_{h>0}$  are fulfilled, we can apply it and conclude that for each for all  $(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \in L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  and for all  $(\psi_h, \tilde{\psi}_h, \tilde{\mathbf{S}}_h)_{h>0} \subset L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  converging strongly to  $(\psi, \tilde{\psi}, \tilde{\mathbf{S}})$  in  $L^p(\Omega^0; \mathbb{R}^3) \times L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  we have

$$\liminf_{h \rightarrow 0} K_h(\psi_h, \tilde{\psi}_h, \tilde{\mathbf{S}}_h) \geq K_0(\psi, \tilde{\psi}, \tilde{\mathbf{S}}). \quad (6.3.2)$$

By taking into account that  $K_h = K_h^{2d, \alpha}$  and  $K_0 = K_0^{2d, \alpha}$ , we conclude that each of four parts of Theorem 6.2.1 the particular functional  $K_0^{2d, \alpha}$  satisfies the "lim inf" part of the definition of the  $\Gamma$ -convergence for the family  $(K_h^{2d, \alpha})_{h>0}$ .

For the "lim sup" part we now have to check if for arbitrary  $(\tilde{\psi}, \tilde{\mathbf{S}}) \in \mathcal{B}_0^{2d, \alpha}$  there exists  $(\tilde{\psi}_h, \tilde{\mathbf{S}}_h)_{h>0} \subset \mathcal{A}^N$  converging strongly to  $(\tilde{\psi}, \tilde{\mathbf{S}})$  in  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  such that  $J_h^{2d, \alpha}(\tilde{\psi}_h, \tilde{\mathbf{S}}_h) \rightarrow J_0^{2d, \alpha}(\tilde{\psi}, \tilde{\mathbf{S}})$  and that the additional condition (4.3.10) is satisfied (in Remark 4.3.2 we established that such sequence, up to the additional condition, exists by the definition of the  $\Gamma$ -convergence of the family  $(J_h)_{h>0}$ ).

Since any such family  $(\tilde{\psi}_h, \tilde{\mathbf{S}}_h)_{h>0}$  converges strongly in  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$ , it is uniformly bounded in the same topology. Also, since it satisfies  $J_h^{2d, \alpha}(\tilde{\psi}_h, \tilde{\mathbf{S}}_h) \rightarrow J_0^{2d, \alpha}(\tilde{\psi}, \tilde{\mathbf{S}})$ , we conclude that values  $J_h^{2d, \alpha}(\tilde{\psi}_h, \tilde{\mathbf{S}}_h)$  are uniformly bounded as well. Thus, from Corollary 6.2.3 we see that this additional condition (4.3.10) is satisfied for all  $\alpha > 0$ .

However, for  $\alpha \in \langle -1, 0 \rangle$  admissible functions  $\tilde{\psi}$  are in  $L^4(\omega; \mathbb{R}^3)$ , so there is no guarantee that a strongly convergent sequence  $(\tilde{\psi}_n, \tilde{\mathbf{S}}_n)_{n>0} \subset \mathcal{A}^N$  (converging to  $(\tilde{\psi}, \tilde{\mathbf{S}})$  in  $L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$ ) satisfies the condition (4.3.10). Thus the part a) of Theorem 6.2.1 is proved in a different way.

Let us take arbitrary  $(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \in \mathcal{A}_0^{2d, \langle -1, 0 \rangle}$ . We prove that for all  $\varepsilon > 0$  there exists  $h^* > 0$  and  $(\psi^*, \tilde{\psi}^*, \tilde{\mathbf{S}}^*) \in \mathcal{A}^{2d, \langle -1, 0 \rangle}$  such that

$$\begin{aligned} \|\psi^* - \psi\|_{L^p(\Omega^0; \mathbb{R}^3)} &\leq \varepsilon, \quad \|\tilde{\psi}^* - \tilde{\psi}\|_{L^4(\omega; \mathbb{R}^3)} \leq \varepsilon, \quad \|\tilde{\mathbf{S}}^* - \tilde{\mathbf{S}}\|_{L^2(\omega; \text{SO}(3))} < \varepsilon \\ \text{and } K_{h^*}^{2d, \langle -1, 0 \rangle}(\psi^*, \tilde{\psi}^*, \tilde{\mathbf{S}}^*) - K_0^{2d, \langle -1, 0 \rangle}(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) &\leq \varepsilon. \end{aligned} \quad (6.3.3)$$

Note that without loss of generality assume that  $\mathbf{f} = 0$ , since the source term is linear and does not affect the convergence. Now we have that  $J_0^{2d, \langle -1, 0 \rangle}(\tilde{\psi}, \tilde{\mathbf{S}}) = 0$  for all  $(\tilde{\psi}, \tilde{\mathbf{S}}) \in L^4(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$ .

Let us take arbitrary  $\varepsilon > 0$ . Firstly, let  $(\psi_m)_{m \geq 1} \subset C^\infty(\Omega^1; \mathbb{R}^3)$  with  $\psi_m|_{x_3=-1} = \text{id}$  be smooth  $W^{1,p}(\Omega^0; \mathbb{R}^3)$  approximations of  $\psi$ , i.e.  $\psi_m \rightarrow \psi$  in  $W^{1,p}(\Omega^0; \mathbb{R}^3)$ . Since  $W^0$

satisfies the growth condition **W0.4**) (and so does  $QW^0$ ), due to the Lebesgue dominated convergence theorem (dominated by  $x \mapsto C(1 + \|\nabla\psi(x)\|_F^p)$ ) we get

$$\Gamma - \mathcal{W}^0(\psi_m) = \int_{\Omega^0} QW^0(\nabla\psi_m)dx \rightarrow \int_{\Omega^0} QW^0(\nabla\psi)dx = \Gamma - \mathcal{W}^0(\psi). \quad (6.3.4)$$

So there is  $m \in \mathbb{N}$  and  $\psi_\infty := \psi_m$  such that

$$\|\psi_\infty - \psi\|_{W^{1,p}(\Omega^0; \mathbb{R}^3)} < \frac{\varepsilon}{2} \text{ and } |(\Gamma - \mathcal{W}^0(\psi_\infty)) - (\Gamma - \mathcal{W}^0(\psi))| < \frac{\varepsilon}{3}. \quad (6.3.5)$$

Due to the trace theorem, we additionally demand that

$$\|\tilde{\psi}_\infty - \tilde{\psi}\|_{L^4(\omega; \mathbb{R}^3)} = \|\psi_\infty - \psi\|_{L^4(\Gamma^0; \mathbb{R}^3)} < \varepsilon, \quad (6.3.6)$$

where  $\tilde{\psi}_\infty := \psi_\infty|_{x_3=0}$ .

As it was announced in Remark 5.4.25, we use Lemma A.1 (instead of Proposition 5.4.24) which claims that there is a sequence  $(\tilde{\psi}_n, \tilde{\mathbf{S}}_n)_{n \geq 1}$  that converges strongly to  $(\tilde{\psi}_\infty, \tilde{\mathbf{S}})$  in  $W^{1,4}(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  so that  $J_{1/n}^{2d, \langle -1, 0 \rangle}(\tilde{\psi}_n, \tilde{\mathbf{S}}_n) \rightarrow J_0^{2d, \langle -1, 0 \rangle}(\tilde{\psi}, \tilde{\mathbf{S}}) = 0$ . Due to strong convergence in  $W^{1,4}(\omega; \mathbb{R}^3)$ , functions  $(\tilde{\psi}_n)_{n \geq 1}$  are uniformly bounded in the same space. Due to the definition of weakly lower semicontinuous envelope, there exists a sequence  $(\psi_n)_{n \geq 1}$  such that  $\psi_n \rightharpoonup \psi_\infty$  in  $W^{1,p}(\Omega^0; \mathbb{R}^3)$  and  $\mathcal{W}^0(\psi_n)dx \rightarrow \Gamma - \mathcal{W}^0(\psi_\infty)$ . Let us apply Lemma 4.3.4 for sequence  $\psi_n \rightarrow \psi_\infty$  and the sequence  $\tilde{\psi}_n \rightarrow \psi_\infty|_{x_3=0}$ . We conclude that there exists  $(\bar{\psi}_n)_{n \geq 1} \subset W^{1,p}(\Omega^0; \mathbb{R}^3)$  such that

$$\bar{\psi}_n \rightarrow \psi_\infty \text{ in } L^p(\Omega^0; \mathbb{R}^3), \bar{\psi}_n|_{x_3=0} = \tilde{\psi}_n \text{ and } \limsup_{n \rightarrow +\infty} \mathcal{W}^0(\bar{\psi}_n) \leq \Gamma - \mathcal{W}^0(\psi_\infty). \quad (6.3.7)$$

We can conclude that there is  $n \in \mathbb{N}$  large enough such that:

$$\begin{aligned} \|\tilde{\mathbf{S}}_n - \tilde{\mathbf{S}}\|_{L^2(\omega; \text{SO}(3))} &< \varepsilon \text{ (due to the definition of the sequence } (\tilde{\psi}_n, \tilde{\mathbf{S}}_n)_{n \geq 1}), \\ \|\bar{\psi}_n - \psi_\infty\|_{L^p(\Omega^0; \mathbb{R}^3)} &< \frac{\varepsilon}{2} \text{ (due to (6.3.7)),} \\ \|\tilde{\psi}_n - \tilde{\psi}_\infty\|_{L^4(\omega; \mathbb{R}^3)} &< \frac{\varepsilon}{2} \text{ (due to the definition of the sequence } (\tilde{\psi}_n, \tilde{\mathbf{S}}_n)_{n \geq 1}), \\ J_{1/n}^{2d, \langle -1, 0 \rangle}(\tilde{\psi}_n, \tilde{\mathbf{S}}_n) &\leq \frac{\varepsilon}{3} \text{ (due to the definition of the sequence} \\ &\quad (\tilde{\psi}_n, \tilde{\mathbf{S}}_n)_{n \geq 1} \text{ and } J_0^{2d, \langle -1, 0 \rangle} \equiv 0), \\ \mathcal{W}^0(\bar{\psi}_n) - (\Gamma - \mathcal{W}^0(\psi_\infty)) &< \frac{\varepsilon}{3} \text{ (due to (6.3.7)).} \end{aligned} \quad (6.3.8)$$

Together with (6.3.5) and (6.3.6), from triangle inequalities we conclude that  $h^* = 1/n$  and  $(\psi^*, \tilde{\psi}^*, \tilde{\mathbf{S}}^*) := (\bar{\psi}_n, \tilde{\psi}_n, \tilde{\mathbf{S}}_n)$  satisfy (6.3.3) and conclude the proof for the **a**) part of Theorem 6.2.1, and the proof for the whole Theorem 6.2.1.  $\square$

*Proof of Theorem 6.2.2.* The proof is done analogously as the proof of Theorem 6.2.1. We use Proposition 4.3.1 in terms of Remark 4.3.3 (both parts 1° and 2°). In the same way all assumptions of the a) part of Proposition 4.3.1 are satisfied. Also, in the same way we conclude that for all  $\alpha \geq 0$  we can apply the b) part of Proposition 4.3.1 (again by Corollary 6.2.3). For the claim of Theorem 6.2.2.a) we can adjust the particular part of proof of Theorem 6.2.1, or we can apply  $\Gamma$ –limit to the inequality

$$K_0^{2d, \langle -1, 0 \rangle}(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \leq K_{\mathbf{S}, h}^{2d, \langle -1, 0 \rangle}(\psi, \tilde{\psi}) \leq K_h^{2d, \langle -1, 0 \rangle}(\psi, \tilde{\psi}, \tilde{\mathbf{S}}), \quad (6.3.9)$$

(which clearly holds for all  $(\psi, \tilde{\psi}, \tilde{\mathbf{S}}) \in \mathcal{A}^{2d}$ ) and conclude the claim.  $\square$

## 6.4. COMPARISON WITH THE 3D–THIN 3D MODEL

At the end, let us compare results from this chapter with results from Chapter 4. Even though some analysis can be done by observing the results from Theorem 6.2.1, we will compare results from Theorem 6.2.2 to the ones from Theorem 4.2.1.

Firstly, we see that in both theorems we have different behaviour of the  $\Gamma$ –limit of particular family of functionals in same distribution of cases:  $\alpha \in \langle -1, 0 \rangle$ ,  $\alpha = 0$ ,  $\alpha \in \langle 0, 5/3 \rangle$ ,  $\alpha = 2$  and  $\alpha \in \langle 2, +\infty \rangle$ . Also, in all cases the sets of admissible functions coincide, and the fact that in the parts a), c) and e) there is no contribution of the thin part to the energy functional in the limit.

The only possible differences are in the energy contributions in cases for  $\alpha = 0$  and  $\alpha = 2$  (i.e. the membrane regime and the flexural regime). However, in those cases firstly, as already said, the sets of admissible functions coincide, and secondly, storage energy functions appearing in those cases ( $Q\tilde{W}_0^1(\nabla'\tilde{\phi})$  and  $Qw_m^0(\nabla'\tilde{\psi})$  for  $\alpha = 0$ , and  $\frac{1}{24}Q_2(\mathbf{b}(\tilde{\phi}))$  and  $\frac{1}{12}\tilde{q}_f(\mathbf{b}(\tilde{\psi}))$  for  $\alpha = 2$ ) have the same form. For  $\alpha = 0$  this is clear, and for  $\alpha = 2$  this is commented in Subsection 5.3.2.

From this we can conclude that there is a possibility that for a particular function  $W_\alpha^1$  one can find functions  $w_m$  and  $q_f$  (satisfying conditions N1), N2) and N3)) such that  $\Gamma$ –limits of functionals  $(K_h^{3d,\alpha})_{h>0}$  and  $(K_{S,h}^{2d,\alpha})_{h>0}$  not only have the same form, but also coincide, for all  $\alpha \in \langle -1, 5/3 \rangle \cup [2, +\infty)$ . We present this in the example of St. Venant–Kirchoff material.

In the literature there are multiple options to define a stored energy functional in the case of St. Venant–Kirchoff material. We present two of them (citeLDRplate, [38]):

$$\begin{aligned}
 W_{STVK,1}(\mathbf{X}) &= \frac{\mu}{4} \operatorname{tr}((\mathbf{X}^T \mathbf{X} - \mathbf{I})^2) + \frac{\lambda}{8} (\operatorname{tr}(\mathbf{X}^T \mathbf{X} - \mathbf{I}))^2, \\
 W_{STVK,2}(\mathbf{X}) &= \begin{cases} \mu \operatorname{tr}((\sqrt{\mathbf{X}^T \mathbf{X}} - \mathbf{I})^2) + \frac{\lambda}{2} (\operatorname{tr}(\sqrt{\mathbf{X}^T \mathbf{X}} - \mathbf{I}))^2, & \det \mathbf{X} > 0, \\ +\infty & \text{otherwise.} \end{cases}
 \end{aligned} \tag{6.4.1}$$

Key reason why we present two options for a stored energy functional is since neither

one of those two functionals satisfy all assumptions W1.1) – W1.3), W2.1) – W2.6) (all assumptions from all cases of Theorem 4.2.1), so there is no one uniform choice for stored energy functional  $W_\alpha^1$  in the case of St. Venant–Kirchoff material. In particular, Le Dret and Raoult in [52] asymptotically found the membrane plate model (Theorem 3.2.1.b)) for the stored energy functional  $W_{STVK,1}$ , and Friesecke, James and Müller asymptotically found the flexural plate model (Theorem 3.2.1.d)) for the stored energy functional  $W_{STVK,2}$  in the case of St. Venant–Kirchoff material.

Thus, to have  $W_\alpha^1$  that satisfies all needed assumptions in Theorem 4.2.1, we will define stored energy function  $W_\alpha^{1,STVK}$  (dependent on  $\alpha$ ) by

$$W_\alpha^{1,STVK}(\mathbf{X}) = \begin{cases} W_{STVK,1}(\mathbf{X}) & \alpha \in \langle -1, 0], \\ W_{STVK,2}(\mathbf{X}) & \alpha \in \langle 0, 5/3 \rangle \cup [2, +\infty). \end{cases} \quad (6.4.2)$$

**Theorem 6.4.1.** Let  $W^0$  be a stored energy function satisfying conditions W0.1)–W0.4). Let  $W_\alpha^1 = W_\alpha^{1,STVK}$  defined in (6.4.2), and  $w_m = w_m^{STVK}$ ,  $q_f = q_f^{STVK}$  defined in (5.2.6). Let those functions define families of functionals  $(K_h^{3d,\alpha})_{h>0}$  and  $(K_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  by (4.1.4) and (6.1.4), respectively.

Then the  $\Gamma$ –limits of families of functionals  $(K_h^{3d,\alpha})_{h>0}$  and  $(K_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  coincide, for all  $\alpha \in \langle -1, 5/3 \rangle \cup [2, +\infty)$ . Consequently, all minimizers of  $\Gamma$ –limits of families of functionals  $(K_h^{3d,\alpha})_{h>0}$  and  $(K_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  coincide

*Proof.* From our reasoning from the beginning of this section, the only thing we have to check is that energy contributions in cases  $\alpha = 0$  and  $\alpha = 2$  in the thin part coincide in  $\Gamma$ –limits  $K_0^{3d,\alpha}$  and  $K_{\mathbf{S},0}^{2d,\alpha}$  from Theorem 4.2.1 and Theorem 6.2.2. This is already done in Proposition 5.4.14 and Remark 5.3.4.  $\square$

**Remark 6.4.2.** Those  $\Gamma$ –limits in this theorem can be written exactly, by using same results we used in the proof above (Theorem 6.2.2, Proposition 5.4.14 and Remark 5.3.4).

However, even though we know explicitly  $\Gamma$ –limits in all cases with respect to  $\alpha \in \langle -1, 5/3 \rangle \cup [2, +\infty)$ , we would like to emphasise that the key point of this theorem is that in all cases those limits coincide. This implies that, for sufficiently small  $h > 0$ , finding minimizers of a functional  $(K_h^{3d,\alpha})_{h>0}$  can be substituted by finding minimizers of  $(K_{\mathbf{S},h}^{2d,\alpha})_{h>0}$  (or even minimizers of  $(K_h^{2d,\alpha})_{h>0}$ ). Thus, instead of solving a problem

defined on a domain that includes a "thin" subset of  $\mathbb{R}^3$  which can be numerically challenging, we can solve a problem defined on a simpler domain, in all cases for  $\alpha$ , meaning that we don't have to choose the parameter  $\alpha$  in advance. This is quite useful since in a real life situation it is unnatural to choose  $\alpha$  (i.e. the magnitude of the energy contribution in the thin part with respect to the thickness of the thin part) in advance, and by this result we can escape that problem.



# CONCLUSION

In this thesis we achieved all goals we announced in the Introduction.

- We rigorously derived limit models for problems describing structures made of the three-dimensional body and the thin layer, with respect to the thickness of the thin layer  $h$  tending to zero, in linear case. We obtained and characterized limits of solutions for all regimes  $\alpha \geq -1$  (Theorem 2.3.4).
- We proposed a linear 3d–2d model which describes the behaviour of the same structure made of two bodies and analysed the asymptotics of that model (when the parameter describing the thickness of the thin layer  $h$  tends to zero). We obtained and characterized limits of solutions for all regimes  $\alpha \geq -1$  (Theorem 2.4.5).
- We compared asymptotics of those two linear models (rigorously derived and our proposed one). We obtained that when the thickness of the layer goes to zero, the solutions of problems characterized by those two models tend to the same limit, in all regimes  $\alpha \geq -1$  (Theorem 2.2.1).
- We proposed a new nonlinear shell model of Naghdi type (in Chapter 5), as a nonlinear generalisation of the linear shell model of Naghdi type from [87]. We investigated and proved some of its properties (full list is in Section 5.1). The most important are that the model is well defined for shells with little regularity ( $W^{1,\infty}(\omega; \mathbb{R}^3)$  parametrisations of the middle surface) and that when  $h \rightarrow 0$ , in certain regimes it tends to rigorously derived membrane [51] and flexural shell model [40] and the constrained membrane plate model [30]. As a matter of fact, we described the asymptotic behavior of the shell in all regimes for  $\alpha \in \langle -1, 5/3 \rangle \cup [2, +\infty)$ , see Theorem 5.4.1 and Theorem 5.4.2.

- We rigorously derived limit models for problems describing structures made of the three-dimensional body and the thin layer, with respect to the thickness of the thin layer  $h$  tending to zero, in nonlinear case as well. We obtained and characterized limit models for all regimes  $\alpha \in \langle -1, 5/3 \rangle \cup [2, +\infty)$  (Theorem 4.2.1).
- We proposed a nonlinear 3d–2d model which describes the behaviour of the same structure made of two bodies and analysed the asymptotics of that model (when the parameter describing the thickness of the thin layer  $h$  tends to zero). We obtained and characterized limit models for all regimes  $\alpha \in \langle -1, 5/3 \rangle \cup [2, +\infty)$  (Theorem 6.2.1 and Theorem 6.2.2).
- We compared asymptotics of those two nonlinear models (rigorously derived and our proposed one). We obtained that when the thickness of the layer goes to zero, the energy functionals characterized by those two models tend to limit models that have the same structure, in all regimes  $\alpha \in \langle -1, 5/3 \rangle \cup [2, +\infty)$  (Chapter 6). More precisely, in the case of the St. Venant–Kirchhoff material, limit models of both nonlinear models coincide in all mentioned regimes, see Theorem 6.4.1.

The analysis in the nonlinear case lacks results in case  $\alpha \in [5/3, 2)$ . However, even simpler problem of rigorous derivation of the plate model in that regime is still an open problem (see [39], [60]), so we can say that our analysis is complete up to today's knowledge.

# APPENDIX

## A. THE MAIN TECHNICAL LEMMA

**Lemma A.1.** Let  $\omega$  be a polygonal domain,  $\boldsymbol{\theta} \in C^2(\bar{\omega}; \mathbb{R}^3)$  such that  $\gamma_0 = \partial\omega$  (i.e. the boundary condition  $\boldsymbol{\psi}|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0}$  holds on the entire boundary). Let there be  $(\boldsymbol{\psi}, \mathbf{S}) \in \mathcal{A}^M$  (the set defined in (5.4.6)). Then there exists sequence  $(\boldsymbol{\psi}_n, \mathbf{S}_n)_{n \geq 1} \subset \mathcal{A}^N$  (the set defined in (5.2.2)) with  $(\boldsymbol{\psi}_n, \mathbf{S}_n) \rightarrow (\boldsymbol{\psi}, \mathbf{S})$  in  $W^{1,4}(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  strongly.

*Proof.* The proof is quite technical and is written in several steps.

The functions  $(\boldsymbol{\psi}_n, \mathbf{S}_n)_{n \geq 1}$  will be defined on a triangular subdivision of the domain  $\omega$  (up to the thin zone near the boundary to adjust the boundary condition for  $\boldsymbol{\psi}_n$ , with which we deal in the last step of the proof). Function  $\boldsymbol{\psi}_n$  will be defined as a piecewise affine function, and  $\mathbf{S}_n$  as a piecewise constant function on a large set, and as a smooth function on a remainder. In this way the value of  $\det \begin{bmatrix} \mathbf{S}_n \mathbf{a}_3 & \partial_1 \boldsymbol{\psi}_n & \partial_2 \boldsymbol{\psi}_n \end{bmatrix}$  will be piecewise constant on a large set and thus easy to control. In the rest of the proof, the pair of functions  $(\boldsymbol{\psi}_\varepsilon, \mathbf{S}_\varepsilon)$ , where  $\boldsymbol{\psi}_\varepsilon$  is piecewise affine and  $\mathbf{S}_\varepsilon$  is piecewise constant we will shortly call piecewise functions.

Also, to make the proof easier to read, we will state the equivalent claim which we will prove: for each  $\varepsilon > 0$  there exists  $(\boldsymbol{\psi}_{n(\varepsilon)}, \mathbf{S}_{n(\varepsilon)}) \in \mathcal{A}^N$  such that  $\|\boldsymbol{\psi}_{n(\varepsilon)} - \boldsymbol{\psi}\|_{W^{1,4}(\omega; \mathbb{R}^3)}^4 + \|\mathbf{S}_{n(\varepsilon)} - \mathbf{S}\|_{L^2(\omega; \text{SO}(3))}^2 < C\varepsilon$ , with  $C > 0$  independent of  $\varepsilon$ .

Often in the proof we will use that a matrix  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$  can be approximated by  $\mathbf{S}_1$  coming from a smaller set. In general there is no guarantee that for (almost) every  $\mathbf{y}' \in \omega$  we have  $\mathbf{S}_1(\mathbf{y}') \in \text{SO}(3)$  (for example when  $\mathbf{S}_1$  is a  $C^\infty$  approximation of  $\mathbf{S}$ ). In that cases we use representation of  $\text{SO}(3)$  matrices via its axial vector  $\mathbf{w} \in \mathbb{R}^3$ :  $\mathbf{S} = \exp(\mathbf{A}_\mathbf{w})$ , where  $\mathbf{A} \in \mathbb{R}^{3 \times 3}$  is skew-symmetric matrix defined as  $\mathbf{A}_\mathbf{w} \mathbf{x} = \mathbf{w} \times \mathbf{x}$ , for all

$\mathbf{x} \in \mathbb{R}^3$ . The map  $\mathbf{w} \mapsto \exp(\mathbf{A}_{\mathbf{w}})$  is infinitely smooth, which can be seen from the Rodrigues' rotation formula (see [76]):

$$\exp(\theta \mathbf{A}_{\mathbf{w}}) = \mathbf{I} + (\sin \theta) \mathbf{A}_{\mathbf{w}} + (1 - \cos \theta) \mathbf{A}_{\mathbf{w}}^2, \quad (0.1.1)$$

for any unit vector  $\mathbf{w}$  and scalar  $\theta$ . On the other hand, if  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$ , then we can choose (almost pointwise)  $\mathbf{w} \in L^2(\omega; B(0, 2\pi))$ .

STEP 1. EXISTENCE OF PIECEWISE APPROXIMATIONS ON REGULAR FAMILY OF SUBDIVISIONS. Here we prove the following: there exists a regular family of subdivisions  $(\mathcal{T}_\varepsilon)_{\varepsilon>0}$  of the domain  $\omega$  such that for each  $\varepsilon > 0$  there is  $\mathcal{T}_\varepsilon \in (\mathcal{T}_\varepsilon)_{\varepsilon>0}$  and piecewise functions  $(\psi_\varepsilon, \mathbf{S}_\varepsilon)$  (with respect to the subdivision  $\mathcal{T}_\varepsilon$ ) such that

$$\|\psi_\varepsilon - \psi\|_{W^{1,4}(\omega; \mathbb{R}^3)}^4 + \|\mathbf{S}_\varepsilon - \mathbf{S}\|_{L^2(\omega; \text{SO}(3))}^2 < \varepsilon. \quad (0.1.2)$$

The proof is a direct consequence of well-known results used in the FEM theory. Let  $(\psi_\infty, \mathbf{S}_\infty) \in C^\infty(\omega; \mathbb{R}^3) \times C^\infty(\omega; \text{SO}(3))$  be  $\varepsilon/2$ -close smooth approximation of functions  $(\psi, \mathbf{S})$  (in  $W^{1,4}(\omega; \mathbb{R}^3) \times L^2(\omega; \text{SO}(3))$  norm). From Theorem 4.4.20. from [17] we have that there exist regular family of subdivisions  $(\mathcal{T}_h)_{h>0}$  and piecewise functions  $(\psi_h, \mathbf{S}_h)_{h>0}$  approximating  $(\psi_\infty, \mathbf{S}_\infty)$ :

$$\begin{aligned} \|\psi_h - \psi_\infty\|_{W^{1,4}(\omega; \mathbb{R}^3)}^4 + \|\mathbf{S}_h - \mathbf{S}_\infty\|_{L^2(\omega; \text{SO}(3))}^2 \\ \leq Ch^2 \left( \|\psi_\infty\|_{W^{2,4}(\omega; \mathbb{R}^3)}^2 + \|\mathbf{S}_\infty\|_{W^{1,2}(\omega; \text{SO}(3))}^2 \right). \end{aligned} \quad (0.1.3)$$

For rotations we actually do as follows. For  $\mathbf{S} \in L^2(\omega; \text{SO}(3))$  we first determine axial vector  $\mathbf{w} \in L^2(\omega; B(0, 2\pi))$ , then make smooth approximation  $\mathbf{w}_\infty \in C^\infty(\omega; B(0, 2\pi))$  and then piecewise constant approximation  $\mathbf{w}_h$ . In this way we are sure that  $\mathbf{S}_h := \exp(\mathbf{A}_{\mathbf{w}_h})$  are indeed in  $\text{SO}(3)$  for all  $\mathbf{y}' \in \omega$ . For  $h$  small enough, the right hand side is smaller than  $\varepsilon/2$ , so we obtain desired inequality after applying the triangle inequality for the subdivision  $\mathcal{T}_\varepsilon = \mathcal{T}_h$  and functions  $\psi_\varepsilon = \psi_h$ ,  $\mathbf{S}_\varepsilon = \mathbf{S}_h$ . Note that the family of subdivisions  $(\mathcal{T}_\varepsilon)_{\varepsilon>0}$  is a subfamily of  $(\mathcal{T}_h)_{h>0}$ , hence it is a regular family.

In the same manner, again by Theorem 4.4.20. from [17], we can find piecewise constant approximation  $\mathbf{a}_3^\varepsilon$  on possibly refined  $\mathcal{T}_\varepsilon$  of the function  $\mathbf{a}_3 \in C^1(\omega; \mathbb{R}^3)$  such that

$$\|\mathbf{a}_3 - \mathbf{a}_3^\varepsilon\|_{L^\infty(\omega; \mathbb{R}^3)} \leq \varepsilon.$$

On refined  $\mathcal{T}_\varepsilon$  the statement (0.1.2) also holds.

STEP 2. INVESTIGATION OF  $\left( \det \begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix} \right)_{\varepsilon > 0}$ .

Let us take the family of subdivisions  $(\mathcal{T}_\varepsilon)_{\varepsilon > 0}$  and the sequence  $(\psi_\varepsilon, \mathbf{S}_\varepsilon)_{\varepsilon > 0}$  of piecewise functions from the last step. Since we have  $\psi_\varepsilon \rightarrow \psi$  in  $W^{1,4}(\omega; \mathbb{R}^3)$  and  $\mathbf{S}_\varepsilon \rightarrow \mathbf{S}$  in  $L^2(\omega; \text{SO}(3))$ , we have that

$$\det \begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix} \rightarrow \det \begin{bmatrix} \mathbf{S} \mathbf{a}_3 & \partial_1 \psi & \partial_2 \psi \end{bmatrix} \text{ in } L^1(\omega; \mathbb{R}). \quad (0.1.4)$$

According to Egorov's theorem, there exists a measurable set  $A \subset \omega$  such that the measure  $|\omega \setminus A|$  is arbitrarily small and such that the upper convergence is uniform on  $A$  a.e. As a consequence of the uniform convergence we have that even if in some point  $\mathbf{y}' \in A$  we have that  $\det \begin{bmatrix} \mathbf{S}_\varepsilon(\mathbf{y}') \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon(\mathbf{y}') & \partial_2 \psi_\varepsilon(\mathbf{y}') \end{bmatrix} < 0$ , this value does not differ by more than some uniform constant  $m$  from  $\det \begin{bmatrix} \mathbf{S}(\mathbf{y}') \mathbf{a}_3 & \partial_1 \psi(\mathbf{y}') & \partial_2 \psi(\mathbf{y}') \end{bmatrix} \geq 0$ , and thus it is larger than  $-m$ . Also, note that the function  $\begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix}$  is piecewise constant on  $\mathcal{T}_\varepsilon$  for each  $\varepsilon > 0$ .

From all this, we can conclude: for each  $\varepsilon > 0$  there exist a subdivision  $\mathcal{T}_\varepsilon$ , piecewise constant  $(\psi_\varepsilon, \mathbf{S}_\varepsilon)$  and a set  $A \subset \omega$  with  $|\omega \setminus A| < \varepsilon$  such that for each triangle  $T_\varepsilon \in \mathcal{T}_\varepsilon$  we are in one of the cases:

1.  $\det \begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix} \geq 0$  on  $T_\varepsilon$ ;
2.  $\det \begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix} < 0$ , and  $T_\varepsilon \setminus (\omega \setminus A)$  is of measure zero;
3.  $\det \begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix} < 0$ , but there exists  $\mathbf{y}' \in T_\varepsilon$  such that

$$\left| \det \begin{bmatrix} \mathbf{S}_\varepsilon(\mathbf{y}') \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon(\mathbf{y}') & \partial_2 \psi_\varepsilon(\mathbf{y}') \end{bmatrix} \right| < m := \varepsilon^5. \quad (0.1.5)$$

Since we have

$$\begin{aligned} \left\| \det \begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix} \right\| &= \left\| \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon \cdot (\partial_1 \psi_\varepsilon \times \partial_2 \psi_\varepsilon) \right\| \\ &= \left\| \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon \right\| \cdot |\cos \angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{a}_3^{\psi_\varepsilon})| \cdot \|\partial_1 \psi_\varepsilon \times \partial_2 \psi_\varepsilon\|, \end{aligned}$$

and since  $\|\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon\| = 1$ , the third case is in the union of the following two:

- 3.1.  $\det \begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix} < 0$ , but there exists  $\mathbf{y}' \in T_\varepsilon$  for which we have the bound  $|\cos \angle(\mathbf{S}_\varepsilon(\mathbf{y}') \mathbf{a}_3^\varepsilon, \mathbf{a}_3^{\psi_\varepsilon(\mathbf{y}')})| < \varepsilon$ ;

3.2.  $\det \begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix} < 0$ , but there exists  $\mathbf{y}' \in T_\varepsilon$  for which we have the bound  $\|\partial_1 \psi_\varepsilon(\mathbf{y}') \times \partial_2 \psi_\varepsilon(\mathbf{y}')\| < \varepsilon^4$ .

STEP 3. EXISTENCE OF PIECEWISE APPROXIMATIONS WITH NONNEGATIVE DETERMINANT. Here we prove the following: for each  $\varepsilon > 0$  there exist a triangular subdivision  $\overline{\mathcal{T}}_\varepsilon$  (not necessarily from the regular family of subdivisions) of the domain  $\omega$  and piecewise functions  $(\overline{\psi}_\varepsilon, \overline{\mathbf{S}}_\varepsilon)$  (with respect to the subdivision  $\overline{\mathcal{T}}_\varepsilon$ ) such that

$$\|\overline{\psi}_\varepsilon - \psi\|_{W^{1,4}(\omega; \mathbb{R}^3)}^4 + \|\overline{\mathbf{S}}_\varepsilon - \mathbf{S}\|_{L^2(\omega; \text{SO}(3))}^2 < C\varepsilon \quad (0.1.6)$$

(with  $C > 0$  independent on  $\varepsilon$ ) and

$$\det \begin{bmatrix} \overline{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \overline{\psi}_\varepsilon & \partial_2 \overline{\psi}_\varepsilon \end{bmatrix} \geq 0. \quad (0.1.7)$$

Let us take subdivision  $\mathcal{T}_\varepsilon$  and pair  $(\psi_\varepsilon, \mathbf{S}_\varepsilon)$  from the end of the last step. We will find subdivision  $\overline{\mathcal{T}}_\varepsilon$  of the subdivision  $\mathcal{T}_\varepsilon$  and piecewise functions  $(\overline{\psi}_\varepsilon, \overline{\mathbf{S}}_\varepsilon)$  (with respect to the subdivision  $\overline{\mathcal{T}}_\varepsilon$ ) close to  $(\psi_\varepsilon, \mathbf{S}_\varepsilon)$  with properties described above.

If a triangle  $T_\varepsilon$  satisfies the case 1 we do not make any changes, i.e.  $T_\varepsilon \in \overline{\mathcal{T}}_\varepsilon$ ,  $(\overline{\psi}_\varepsilon, \overline{\mathbf{S}}_\varepsilon) := (\psi_\varepsilon, \mathbf{S}_\varepsilon)$ , and no error is made.

If a triangle  $T_\varepsilon$  satisfies the case 2 we make change only on  $\mathbf{S}_\varepsilon$ , i.e.  $T_\varepsilon \in \overline{\mathcal{T}}_\varepsilon$ ,  $\overline{\psi}_\varepsilon = \psi_\varepsilon$  and  $\overline{\mathbf{S}}_\varepsilon = -\mathbf{S}_\varepsilon$  on  $T_\varepsilon$ . The error we make on such triangles is only in rotations and it is bounded by

$$\|\mathbf{S}_\varepsilon - \overline{\mathbf{S}}_\varepsilon\|_{L^2(\omega; \text{SO}(3))}^2 \leq 2|(\omega \setminus A)| \max_{\mathbf{R} \in \text{SO}(3)} \|\mathbf{R}\|_F \leq C\varepsilon. \quad (0.1.8)$$

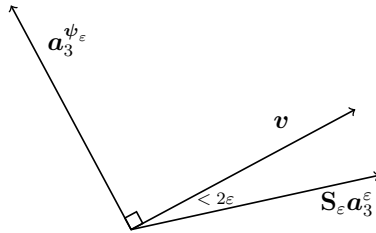


Figure A.1: Sketch for the case 3.1. Vector  $\mathbf{v}$  is the unit vector in the plane spanned by  $\mathbf{a}_3^{\psi_\varepsilon}$  and  $\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon$  perpendicular on  $\mathbf{a}_3^{\psi_\varepsilon}$ . The matrix  $\mathbf{R}$  rotates  $\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon$  to  $\mathbf{v}$  about vector  $\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon \times \mathbf{v}$ .

Let now triangle  $T_\varepsilon$  belongs to the case 3.1 and not the case 3.2. Since we are not in the case 3.2  $\mathbf{a}_3^{\psi_\varepsilon}$  exists. The idea is to rotate  $\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon$  such that it becomes orthogonal

to  $\mathbf{a}_3^{\psi_\varepsilon}$  and thus the determinant will become zero. This rotation will be denoted by  $\mathbf{R}$ . Since  $\det \begin{bmatrix} \mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \psi_\varepsilon & \partial_2 \psi_\varepsilon \end{bmatrix} < 0$  we actually have  $-\varepsilon < \cos \angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{a}_3^\psi) < 0$ . Thus  $\angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{a}_3^\psi) > \pi/2$  and

$$-\varepsilon < \cos \angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{a}_3^\psi) = \sin \left( \frac{\pi}{2} - \angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{a}_3^\psi) \right) = -\sin \left( \angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{a}_3^\psi) - \frac{\pi}{2} \right).$$

From  $\sin t \approx t$  near  $t = 0$ , we can bound  $\sin t > t/2$  for sufficiently small  $t > 0$ . Using that bound in above inequality, we obtain

$$\varepsilon > \sin \left( \angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{a}_3^\psi) - \frac{\pi}{2} \right) > \frac{1}{2} \left( \angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{a}_3^\psi) - \frac{\pi}{2} \right),$$

so we conclude  $\pi/2 < \angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{a}_3^{\psi_\varepsilon}) < \pi/2 + 2\varepsilon$ .

Let us now define vector  $\mathbf{v} \in \mathbb{R}^3$  as  $\mathbf{v} = \mathbf{a}_3^{\psi_\varepsilon} \times (\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon \times \mathbf{a}_3^{\psi_\varepsilon})$  (see Figure A.1). It is the unit vector that lies in the plane spanned by  $\mathbf{a}_3^{\psi_\varepsilon}$  and  $\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon$  and it is perpendicular to  $\mathbf{a}_3^{\psi_\varepsilon}$ . Also, we have  $\alpha := \angle(\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon, \mathbf{v}) < 2\varepsilon$ . Let us define  $\mathbf{R} = \exp(\alpha \mathbf{A}_{\mathbf{S}_\varepsilon \mathbf{a}_3^\varepsilon \times \mathbf{v}})$ . Now we can define  $\bar{T}_\varepsilon = T_\varepsilon$ ,  $\bar{\psi}_\varepsilon = \psi_\varepsilon$  and  $\bar{\mathbf{S}}_\varepsilon = \mathbf{R} \mathbf{S}_\varepsilon$ . From geometrical properties, we have that  $\bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon = \mathbf{v}$  and hence perpendicular to  $\mathbf{a}_3^{\bar{\psi}_\varepsilon}$ , so we have  $\det \begin{bmatrix} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \bar{\psi}_\varepsilon & \partial_2 \bar{\psi}_\varepsilon \end{bmatrix} = 0$ .

The approximation error is made only in rotation matrices and it is estimated pointwise by

$$\|\mathbf{S}_\varepsilon - \bar{\mathbf{S}}_\varepsilon\|_F \leq \|\mathbf{S}_\varepsilon\|_F \cdot \|\mathbf{I} - \mathbf{R}\|_F \leq C\varepsilon \quad (0.1.9)$$

since  $\mathbf{R}$  is a rotation for an angle less than  $\varepsilon$  by Rodrigues' formula (0.1.1). The whole error is then bounded by

$$\sum_{\bar{T}_\varepsilon \text{ in case 3.1 and not in 3.2}} \|\mathbf{S}_\varepsilon - \bar{\mathbf{S}}_\varepsilon\|_{L^2(\bar{T}_\varepsilon; \text{SO}(3))}^2 \leq C\varepsilon |\omega| \leq C\varepsilon. \quad (0.1.10)$$

Lastly, let us take the most complicated case 3.2. Let us firstly prove that condition  $\|\partial_1 \psi_\varepsilon \times \partial_2 \psi_\varepsilon\| < \varepsilon^4$  implies that there is unit  $\mathbf{w} \in \mathbb{R}^2$  such that  $\|D_{\mathbf{w}} \psi_\varepsilon\| < \varepsilon^2$ . Since  $\psi_\varepsilon$  is affine function on  $T_\varepsilon$  we can define  $\phi$  as the same affine function defined on the unit square  $[0, 1]^2$ , see Figure A.2.

Since  $\phi$  is affine, its image is parallelogram with sides of length  $\|\partial_1 \phi\|$  and  $\|\partial_2 \phi\|$  and area  $\|\partial_1 \phi \times \partial_2 \phi\| < \varepsilon^4$ . If any of its sides  $\|\partial_1 \phi\|$ ,  $\|\partial_2 \phi\|$  is less than  $\varepsilon^2$ , the proof is done ( $\mathbf{w} = \mathbf{e}_1$  or  $\mathbf{w} = \mathbf{e}_2$ ). Otherwise, the length of height  $\mathbf{h}$  of the parallelogram satisfies

$$\|\mathbf{h}\| = \frac{A}{\|\partial_1 \phi\|} = \frac{\|\partial_1 \phi \times \partial_2 \phi\|}{\|\partial_1 \phi\|} < \varepsilon^2. \quad (0.1.11)$$

The inverse image of any height  $\phi^{-1}(\mathbf{h})$  is a line in unit square with the length greater than 1. The derivative of  $\phi$  in the direction along this line is the ratio of the length of the height  $\mathbf{h}$  and the length of preimage of the height  $\phi^{-1}(\mathbf{h})$ , and it is less than  $\varepsilon^2$ . Thus  $\mathbf{w}$  is the unit vector determined by  $\phi^{-1}(\mathbf{h})$ .

We will construct  $\overline{T}_\varepsilon$  as a subdivision of  $T_\varepsilon$ . To be more precise we will split it into seven triangles. Let us take that  $T_\varepsilon$ . It is a triangle with vertices  $A$ ,  $B$  and  $C$ . Let the triangle  $A'B'C'$  be obtained by the homothety of the triangle  $ABC$  in respect of the incenter of the triangle  $ABC$  (which we denote by  $I$ ) with the coefficient  $(1 - \varepsilon)$  (see Figure A.3). New subdivision in  $\overline{T}_\varepsilon$ , instead  $T_\varepsilon \equiv ABC$ , form triangles  $A'B'C'$ ,  $AA'C$ ,  $A'CC'$ ,  $BA'B'$ ,  $ABA'$ ,  $C'CB'$  and  $BB'C$ .

Let us take unit vector  $\mathbf{w} \in \mathbb{R}^2$  such that  $\|D_{\mathbf{w}}\psi_\varepsilon\| < \varepsilon^2$  (which exists by the above argument). Let us take a line parallel to this vector such that it passes through one of the vertices  $A'$ ,  $B'$  or  $C'$  and intersects the closed triangle  $A'B'C'$ . Without loss of generality it passes through  $A'$ . Let us introduce  $D'$  as the intersection of that line and side  $\overline{B'C'}$ . Note that in the case that  $\mathbf{w}$  is parallel to a side of the triangle,  $D'$  coincides with one of the points  $B'$ ,  $C'$ .

Let us define piecewise affine function  $\eta$  on the triangle  $T_\varepsilon$  (with respect to the described subdivision of the triangle) such that  $\eta$  is equal to zero in vertices  $A$ ,  $B$ ,  $C$ ,  $B'$  and  $C'$ , and such that  $\eta(A') = \psi_\varepsilon(D') - \psi_\varepsilon(A')$ . Then we define  $\overline{\psi}_\varepsilon$  on  $\overline{T}_\varepsilon$  by  $\overline{\psi}_\varepsilon = \psi_\varepsilon + \eta$

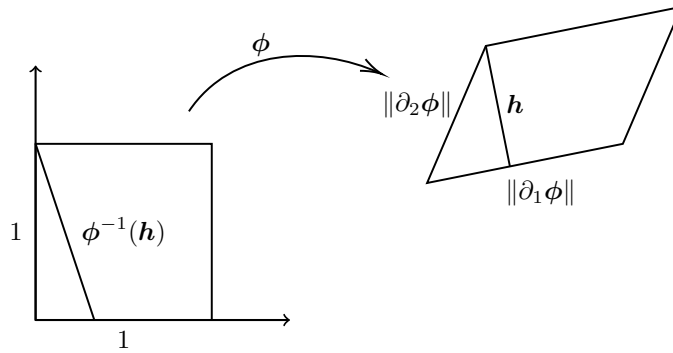


Figure A.2: Sketch for the proof of existence of vector  $\mathbf{w}$ .



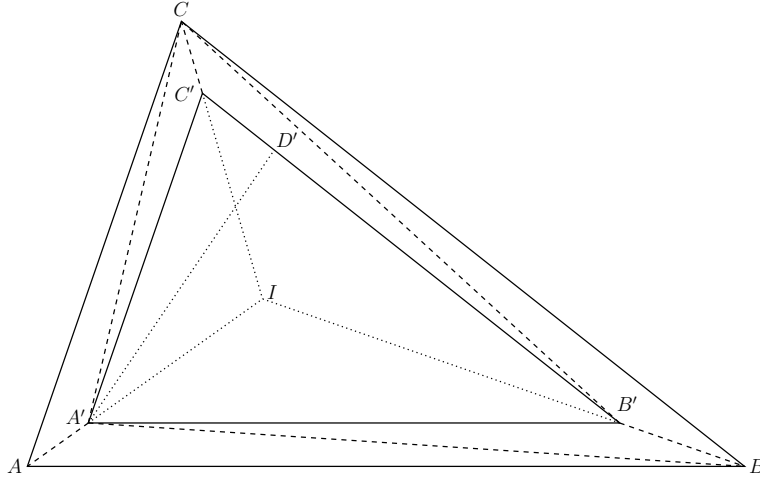


Figure A.3: Triangulation of the triangle  $ABC$ . Needed in case 3.2, and in Step 4.

and

$$\bar{\mathbf{S}}_\varepsilon = \begin{cases} \mathbf{S}_\varepsilon & \mathbf{y}' \in A'B'C', \\ \text{"anything allowed"} & \text{otherwise.} \end{cases} \quad (0.1.12)$$

More precisely, on each triangle from the subdivision except  $A'B'C'$  we define  $\bar{\mathbf{S}}_\varepsilon$  to take any value such that  $\det \begin{bmatrix} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \bar{\psi}_\varepsilon & \partial_2 \bar{\psi}_\varepsilon \end{bmatrix} \geq 0$  on each triangle. Since (from the construction)  $D_w \bar{\psi}_\varepsilon = \frac{\bar{\psi}_\varepsilon(D') - \bar{\psi}_\varepsilon(A')}{|D'A'|} = 0$ , vectors  $\partial_1 \bar{\psi}_\varepsilon$  and  $\partial_2 \bar{\psi}_\varepsilon$  are linear dependent, and thus  $\|\partial_1 \bar{\psi}_\varepsilon \times \partial_2 \bar{\psi}_\varepsilon\| = 0$  and  $\det \begin{bmatrix} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \bar{\psi}_\varepsilon & \partial_2 \bar{\psi}_\varepsilon \end{bmatrix} = 0$  on triangle  $A'B'C'$ . Thus for this choice for  $(\bar{\psi}_\varepsilon, \bar{\mathbf{S}}_\varepsilon)$  we have that  $\det \begin{bmatrix} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \bar{\psi}_\varepsilon & \partial_2 \bar{\psi}_\varepsilon \end{bmatrix} \geq 0$  on the whole  $T_\varepsilon$ .

Since  $\bar{\mathbf{S}}_\varepsilon \neq \mathbf{S}_\varepsilon$  only on  $ABC \setminus A'B'C'$ , and since the coefficient of the homothety is  $(1 - \varepsilon)$ , the area on which error is made is of order of  $\varepsilon$  and we bound the error as in case 2, i.e. (0.1.10)

Before we proceed, let us recall that the family of subdivisions  $(\mathcal{T}_\varepsilon)_{\varepsilon>0}$  is a regular family, which can be stated as following statement: there exist  $0 < \theta_1 < \theta_2$  (independent of  $\varepsilon$ ) such that all angles in triangles  $T_\varepsilon$  are bounded from below and above by  $\theta_1$  and  $\theta_2$ , respectively. From the formula of the area of the triangle  $A_{T_\varepsilon} = \frac{1}{2}ab \sin \gamma$ , we can also prove the following: there exist constants  $c_1, c_2$  independent of  $\varepsilon$  such that the length of each side of the triangle  $T_\varepsilon$  is in the interval  $[c_1 \sqrt{A_{T_\varepsilon}}, c_2 \sqrt{A_{T_\varepsilon}}]$ .

The error for  $\bar{\psi}_\varepsilon$  depends entirely on the function  $\eta$ . From the definition of the func-

tion, we have

$$\|\boldsymbol{\eta}(A')\| = \|\boldsymbol{\psi}_\varepsilon(D') - \boldsymbol{\psi}_\varepsilon(A')\| \leq \|D_{\mathbf{w}}\boldsymbol{\psi}_\varepsilon\| \cdot |D'A'|. \quad (0.1.13)$$

For  $|D'A'|$  we have

$$|D'A'| \leq \max\{|A'C'|, |A'B'|\} = (1 - \varepsilon) \max\{|AC|, |AB|\} \leq c_2 \sqrt{A_{T_\varepsilon}}, \quad (0.1.14)$$

so we can bound

$$\|\boldsymbol{\eta}(A')\| \leq c_2 \sqrt{A_{T_\varepsilon}} \varepsilon^2. \quad (0.1.15)$$

From that we firstly have

$$\sum_{\bar{T}_\varepsilon \text{ in case 3.2}} \|\bar{\boldsymbol{\psi}}_\varepsilon - \boldsymbol{\psi}_\varepsilon\|_{L^2(\bar{T}_\varepsilon; \mathbb{R}^3)}^4 \leq \sum_{T_\varepsilon \in \mathcal{T}_\varepsilon} A_{T_\varepsilon} c_2^4 A_{T_\varepsilon}^2 \varepsilon^8 \leq c_2^4 |\omega|^2 \varepsilon^8 \sum_{T_\varepsilon \in \mathcal{T}_\varepsilon} A_{T_\varepsilon} \leq C \varepsilon^8. \quad (0.1.16)$$

Now we estimate the derivatives. Function  $\boldsymbol{\eta}$  has derivative equal to zero on all triangles which do not contain  $A'$  as a vertex. Additionally, on those triangles in which  $A'$  is a vertex, the value of  $\boldsymbol{\eta}$  is equal to 0 in two other vertices, so the derivative in the direction parallel to the side determined by those two vertices is zero. To bound  $\nabla' \boldsymbol{\eta}$ , we have to determine the derivative in the direction perpendicular to that side. Since the function is affine, for all triangles of the form  $A'XY$  this derivative is of the form

$$\frac{\boldsymbol{\eta}(A') - 0}{\text{dist}(A', XY)}. \quad (0.1.17)$$

Let us bound the denominator from below for our 5 triangles in  $T_\varepsilon$  containing  $A'$ :

- $A'XY \equiv A'AC$  (for  $A'AB$  similarly):

$$\begin{aligned} \text{dist}(A', AC) &= \varepsilon \text{dist}(I, AC) = \varepsilon \frac{1}{2} |AC| \tan \alpha / 2 \\ &\geq \varepsilon \frac{1}{2} c_1 \sqrt{A_{T_\varepsilon}} \tan \theta_1 / 2 = c \varepsilon \sqrt{A_{T_\varepsilon}}, \end{aligned} \quad (0.1.18)$$

where  $c$  is independent of  $\varepsilon$  and the triangle  $T_\varepsilon$ .

- $A'XY \equiv A'CC'$  (for  $A'BB'$  similarly):

$$\begin{aligned} \text{dist}(A', CC') &= (1 - \varepsilon) \text{dist}(A, CI) = (1 - \varepsilon) |AC| \sin \gamma / 2 \\ &\geq \frac{1}{2} c_1 \sqrt{A_{T_\varepsilon}} \sin \theta_1 / 2 = c \sqrt{A_{T_\varepsilon}}, \end{aligned} \quad (0.1.19)$$

where we additionally used that without loss of generality  $\varepsilon \leq 1/2$ .

- $A'XY \equiv A'BC'$ :

$$\text{dist}(A', B'C') = (1 - \varepsilon) \text{dist}(A, BC) = (1 - \varepsilon) \frac{2A_{T_\varepsilon}}{|BC|} \geq \frac{2A_{T_\varepsilon}}{c_2 \sqrt{A_{T_\varepsilon}}} = c \sqrt{A_{T_\varepsilon}}. \quad (0.1.20)$$

In worst case scenario, the bound is of the form  $c\varepsilon \sqrt{A_{T_\varepsilon}}$ . Now we can calculate for each triangle of the form  $A'XY$  using (0.1.15)

$$\begin{aligned} \|\nabla' \bar{\psi}_\varepsilon - \nabla' \psi_\varepsilon\|_{L^4(A'XY; \mathbb{R}^{3 \times 2})}^4 &= \|\nabla' \boldsymbol{\eta}\|_{L^4(A'XY; \mathbb{R}^{3 \times 2})}^4 = \|\boldsymbol{\eta}(A')\|^4 \text{dist}(A', XY)^{-4} A_{A'XY} \\ &\leq \left( \frac{c_2 \sqrt{A_{T_\varepsilon}} \varepsilon^2}{c\varepsilon \sqrt{A_{T_\varepsilon}}} \right)^4 A_{A'XY} \leq C\varepsilon^4 A_{A'XY}. \end{aligned} \quad (0.1.21)$$

After summing up over all 7 triangles in  $T_\varepsilon$  we obtain that for  $\bar{T}_\varepsilon$  from the case 3.2 we have

$$\|\bar{\psi}_\varepsilon - \psi_\varepsilon\|_{W^{1,4}(\bar{T}_\varepsilon; \mathbb{R}^3)}^4 \leq C\varepsilon^4.$$

After summing estimates over all triangles in  $\mathcal{T}_\varepsilon$ , we obtain

$$\|\nabla' \bar{\psi}_\varepsilon - \nabla' \psi_\varepsilon\|_{L^4(\omega; \mathbb{R}^{3 \times 2})}^4 \leq C\varepsilon^4 |\omega| \leq C\varepsilon^4, \quad (0.1.22)$$

so, specially, the  $W^{1,4}(\omega; \mathbb{R}^3)$  error between  $\psi_\varepsilon$  and  $\bar{\psi}_\varepsilon$  is bounded by  $C\varepsilon$ .

Since in the all possibilities described in the end of the Step 2. we have that error between pair of functions  $(\psi_\varepsilon, \mathbf{S}_\varepsilon)$  and  $(\bar{\psi}_\varepsilon, \bar{\mathbf{S}}_\varepsilon)$  is bounded by  $C\varepsilon$ , we obtain that the claim for this step is proven.

STEP 4. CHANGE  $a_3^\varepsilon$  BY  $a_3$ . Until now we have proved that on each triangle  $\bar{T}_\varepsilon$  the inequality  $\det \begin{bmatrix} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon & \partial_1 \bar{\psi}_\varepsilon & \partial_2 \bar{\psi}_\varepsilon \end{bmatrix} \geq 0$  holds. We want to replace  $\mathbf{a}_3^\varepsilon$  by  $\mathbf{a}_3$  and if necessary we will also change values of  $\bar{\mathbf{S}}_\varepsilon$  so that we obtain inequality

$$\det \begin{bmatrix} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3(\mathbf{y}') & \partial_1 \bar{\psi}_\varepsilon & \partial_2 \bar{\psi}_\varepsilon \end{bmatrix} \geq 0,$$

for each  $\mathbf{y}' \in \omega$ . This step will be similar to the case 3.1. of the previous step.

Let us take a triangle  $\bar{T}_\varepsilon$  for which there exists a point  $\mathbf{y}' \in \bar{T}_\varepsilon$  having property  $\det \begin{bmatrix} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3(\mathbf{y}') & \partial_1 \bar{\psi}_\varepsilon & \partial_2 \bar{\psi}_\varepsilon \end{bmatrix} < 0$ . Specially, vectors  $\partial_1 \bar{\psi}_\varepsilon$  and  $\partial_2 \bar{\psi}_\varepsilon$  are linearly independent and thus  $\mathbf{a}_3^{\bar{\psi}_\varepsilon}$  exists. Note that for such (fixed)  $\partial_1 \bar{\psi}_\varepsilon$  and  $\partial_2 \bar{\psi}_\varepsilon$  the set of unit vectors  $\mathbf{v} \in \mathbb{R}^3$  such that the inequality  $\det \begin{bmatrix} \mathbf{v} & \partial_1 \bar{\psi}_\varepsilon & \partial_2 \bar{\psi}_\varepsilon \end{bmatrix} \geq 0$  is satisfied is a hemisphere with a pole at vector  $\mathbf{a}_3^{\bar{\psi}_\varepsilon}$ . Conditions from this step then imply that  $\angle(\mathbf{a}_3^{\bar{\psi}_\varepsilon}, \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon) \leq \pi/2$

and  $\angle(\bar{\mathbf{a}}_3^{\bar{\psi}_\varepsilon}, \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3(\mathbf{y}')) > \pi/2$ . Since  $\|\mathbf{a}_3(\mathbf{y}') - \mathbf{a}_3^\varepsilon\| \leq \varepsilon$ , similarly to the case 3.1. from the last step (due to  $\sin t \approx t$ ) we can conclude that  $\angle(\bar{\mathbf{S}}_\varepsilon \mathbf{a}_3(\mathbf{y}'), \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon) = \angle(\mathbf{a}_3(\mathbf{y}'), \mathbf{a}_3^\varepsilon) \leq 2\varepsilon$ .

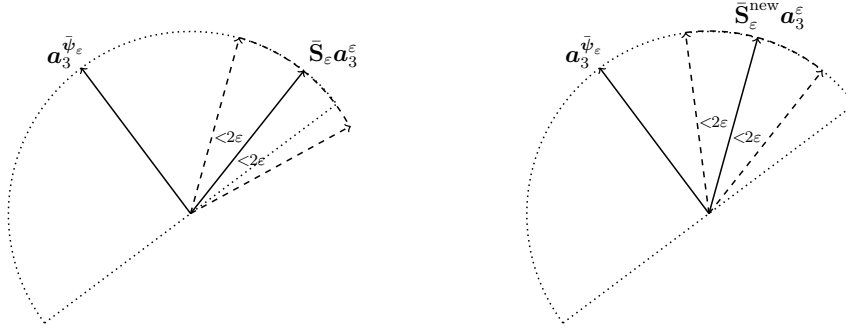


Figure A.4: The idea behind the Step 4. The hemisphere with allowed vectors  $\mathbf{v}$  such that  $\det[\mathbf{v} \partial_1 \bar{\psi}_\varepsilon \partial_2 \bar{\psi}_\varepsilon] \geq 0$  is dotted, only in the plane spanned by  $\mathbf{a}_3^{\bar{\psi}_\varepsilon}$  and  $\bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon$ . All vectors  $\bar{\mathbf{S}}_\varepsilon \mathbf{a}_3(\mathbf{y}')$  are  $2\varepsilon$ -close to  $\bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon$  (note that they don't have to be in the same plane as the described one), and are plotted dashed. By rotating the vector  $\bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon$  for an angle of  $2\varepsilon$  closer to  $\mathbf{a}_3^{\bar{\psi}_\varepsilon}$ , all vectors  $\bar{\mathbf{S}}_\varepsilon^{\text{new}} \mathbf{a}_3(\mathbf{y}')$  are now in the allowed hemisphere.

Now we change the value of  $\bar{\mathbf{S}}_\varepsilon$  on  $\bar{T}_\varepsilon$  to the value  $\bar{\mathbf{S}}_\varepsilon^{\text{new}} := \mathbf{R} \bar{\mathbf{S}}_\varepsilon$ , where  $\mathbf{R} := \exp(2\varepsilon \mathbf{A}_{\bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon \times \mathbf{a}_3^{\bar{\psi}_\varepsilon}})$ , see Figure A.4. The matrix  $\mathbf{R}$  rotates  $\bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon$  closer to  $\mathbf{a}_3^{\bar{\psi}_\varepsilon}$ , such that  $\angle(\mathbf{a}_3^{\bar{\psi}_\varepsilon}, \mathbf{R} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon) \leq \pi/2 - 2\varepsilon$ . That implies that for all  $\mathbf{y}' \in \bar{T}_\varepsilon$

$$\angle(\mathbf{a}_3^{\bar{\psi}_\varepsilon}, \mathbf{R} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3(\mathbf{y}')) \leq \angle(\mathbf{a}_3^{\bar{\psi}_\varepsilon}, \mathbf{R} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon) + \angle(\mathbf{R} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3(\mathbf{y}'), \mathbf{R} \bar{\mathbf{S}}_\varepsilon \mathbf{a}_3^\varepsilon) \leq \pi/2, \quad (0.1.23)$$

so as the consequence we have that  $\det[\bar{\mathbf{S}}_\varepsilon^{\text{new}} \mathbf{a}_3^\varepsilon \partial_1 \bar{\psi}_\varepsilon \partial_2 \bar{\psi}_\varepsilon] \geq 0$ . The error  $\|\bar{\mathbf{S}}_\varepsilon^{\text{new}} - \bar{\mathbf{S}}_\varepsilon\|_{L^2(\omega; \text{SO}(3))}^2$  is again bounded by  $C\varepsilon$ , since the matrix  $\mathbf{R}$  is a rotation for an angle of order  $\varepsilon$ . The same rotation holds for all  $\mathbf{y}' \in \bar{T}_\varepsilon$ .

To simplify the notation in the rest of the proof we use the notation  $\bar{\mathbf{S}}_\varepsilon$  for  $\bar{\mathbf{S}}_\varepsilon^{\text{new}}$ .

**STEP 5. SMOOTHENING  $\bar{\mathbf{S}}_\varepsilon$ .** The only criteria that is not satisfied to conclude the proof apart from the boundary conditions is the regularity of  $\bar{\mathbf{S}}_\varepsilon$ : our approximations  $\bar{\mathbf{S}}_\varepsilon$  are piecewise constant, which is not in  $W^{1,2}(\omega; \text{SO}(3))$ . For that we proceed with one more step.

Let us take piecewise functions  $(\bar{\psi}_\varepsilon, \bar{\mathbf{S}}_\varepsilon)$  and their triangular subdivision  $\bar{\mathcal{T}}_\varepsilon$  from the last step. We will construct  $(\bar{\bar{\psi}}_\varepsilon, \bar{\bar{\mathbf{S}}}_\varepsilon)$  and  $\bar{\bar{\mathcal{T}}}_\varepsilon$  with  $\bar{\bar{\psi}}_\varepsilon$  being piecewise affine arbitrarily close to  $\bar{\psi}_\varepsilon$  in  $W^{1,4}(\omega; \mathbb{R}^3)$  norm and  $\bar{\bar{\mathbf{S}}}_\varepsilon$  being continuous and arbitrarily close to  $\bar{\mathbf{S}}_\varepsilon$  in  $L^2(\omega; \text{SO}(3))$  norm. The subdivision  $\bar{\bar{\mathcal{T}}}_\varepsilon$  will be subdivision of  $\bar{\mathcal{T}}_\varepsilon$ . This will conclude the proof.

Let us take any  $\delta > 0$  and any triangle  $T_\varepsilon \in \bar{\mathcal{T}}_\varepsilon$ . Let us perform the same subdivision of the triangle  $T_\varepsilon$  as in the last case of previous step (see Figure A.3): for triangle  $T_\varepsilon \equiv ABC$  we will find its homothety image  $A'B'C'$  with respect to its incenter  $I$  with coefficient  $(1 - \delta)$ , and then in subdivision  $\bar{\bar{\mathcal{T}}}_\varepsilon$  we take triangles  $A'B'C'$ ,  $AA'C'$ ,  $A'CC'$ ,  $BAB'$ ,  $ABA'$ ,  $C'CB'$  and  $BB'C'$ .

On this subdivision we define  $\bar{\bar{\psi}}_\varepsilon$  as piecewise affine function (with respect to the described subdivision) with values  $\bar{\bar{\psi}}_\varepsilon(A) = \bar{\bar{\psi}}_\varepsilon(A') = \bar{\psi}_\varepsilon(A)$ ,  $\bar{\bar{\psi}}_\varepsilon(B) = \bar{\bar{\psi}}_\varepsilon(B') = \bar{\psi}_\varepsilon(B)$  and  $\bar{\bar{\psi}}_\varepsilon(C) = \bar{\bar{\psi}}_\varepsilon(C') = \bar{\psi}_\varepsilon(C)$ . For the rotation we define  $\bar{\bar{\mathbf{S}}}_\varepsilon = \exp(\xi \mathbf{A}_w)$ , where  $w$  is the axial vector of the matrix  $\bar{\mathbf{S}}_\varepsilon$ , and  $\xi$  smooth function with values in  $[0, 1]$  such that it is equal to 1 on  $A'B'C'$  and 0 on the boundary of the triangle  $ABC$ . Due to Rodrigues' formula (0.1.1), function  $\bar{\bar{\mathbf{S}}}_\varepsilon$  is smooth on  $\bar{\bar{\mathcal{T}}}_\varepsilon$ .

By the definition,  $\bar{\bar{\psi}}_\varepsilon$  is piecewise affine,  $\bar{\bar{\mathbf{S}}}_\varepsilon$  is continuous on the whole domain  $\omega$  (since on the boundary of all triangles from  $\bar{\mathcal{T}}_\varepsilon$  its value is  $\mathbf{I}$ ).

We still have the condition  $\det \begin{bmatrix} \bar{\bar{\mathbf{S}}}_\varepsilon \mathbf{a}_3 & \partial_1 \bar{\bar{\psi}}_\varepsilon & \partial_2 \bar{\bar{\psi}}_\varepsilon \end{bmatrix} \geq 0$  is satisfied: on the triangle  $A'B'C'$  due to properties of functions  $\bar{\bar{\psi}}_\varepsilon$  and  $\bar{\bar{\mathbf{S}}}_\varepsilon$ , on other triangles due to fact that  $\|\partial_1 \bar{\bar{\psi}}_\varepsilon \times \partial_2 \bar{\bar{\psi}}_\varepsilon\| = 0$  (along lines  $\overline{AA'}$ ,  $\overline{BB'}$ ,  $\overline{CC'}$  function  $\bar{\bar{\psi}}_\varepsilon$  is constant). Thus we have  $(\bar{\bar{\psi}}_\varepsilon, \bar{\bar{\mathbf{S}}}_\varepsilon) \in \mathcal{A}^M$ . On the other hand, due to construction, the error

$$\|\bar{\bar{\psi}}_\varepsilon - \bar{\psi}_\varepsilon\|_{W^{1,4}(\omega; \mathbb{R}^3)}^2 + \|\bar{\bar{\mathbf{S}}}_\varepsilon - \bar{\mathbf{S}}_\varepsilon\|_{L^2(\omega; \text{SO}(3))}^2 \quad (0.1.24)$$

goes to zero as  $\delta \rightarrow 0$ , so it can be arbitrarily small.

**STEP 6. BOUNDARY CONDITIONS.** First note that the Dirichlet boundary condition for  $\bar{\bar{\mathbf{S}}}_\varepsilon$  is trivially satisfied since  $\bar{\bar{\mathbf{S}}}_\varepsilon = \mathbf{I}$  near the boundary of each triangle.

Also note that if  $\theta|_{\gamma_0}$  is piecewise affine function (with respect to the boundary of polygonal domain  $\omega$ ), that the Dirichlet boundary condition for  $\bar{\bar{\psi}}_\varepsilon$  is satisfied as well: they are satisfied in the Step 1 of the proof (when the FEM approximation is made) and all steps that came after did not change value of  $\psi_\varepsilon / \bar{\psi}_\varepsilon / \bar{\bar{\psi}}_\varepsilon$  on the boundary. If this is

not the case, we have to make one additional approximation.

Let us firstly take  $\psi_\infty$ , a smooth approximation of  $\psi$  on  $\bar{\omega}$  in  $W^{1,4}(\omega; \mathbb{R}^3)$  norm that satisfies the Dirichlet boundary conditions from  $\mathcal{A}^N$ . The value of  $\psi_\infty$  and its derivatives are bounded in  $L^\infty(\omega; \mathbb{R})$  norm.

Let  $\delta > 0$  be a small parameter. Let us find polygonal lines in  $\omega$  with sides parallel to sides of  $\partial\omega$  which are exactly  $\delta$  and  $2\delta$  close to the boundary. The line closer to  $\ell_0 := \partial\omega$  we will call  $\ell_1$  and the other one  $\ell_2$ . The interior of  $\ell_2$  will be  $\omega'$ . Buffer zones between lines  $\ell_i$  and  $\ell_j$ ,  $i, j \in \{0, 1, 2\}$  will be denoted by  $[\ell_i, \ell_j]$ . See Figure A.5.

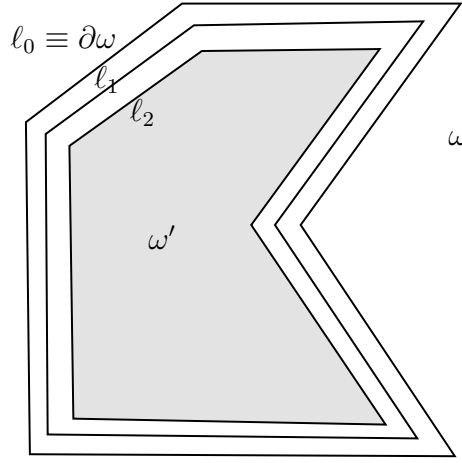


Figure A.5: The sketch of the partition of the domain  $\omega$  in Step 6.

Lines  $\ell_0, \ell_1, \ell_2$  are made of sides (the ones that are parallel to each other) and corresponding vertices (those which are ending points of corresponding sides). Two corresponding (parallel) sides in  $\ell_0$  and  $\ell_1$  (and in  $\ell_1$  and  $\ell_2$  as well) together with their four endpoints define a thin and long trapezoid with height  $\delta$ . Angles in those trapezoids depend only on the geometry of  $\partial\omega$  and do not depend on  $\delta$  (vertices on  $\ell_1$  and  $\ell_2$  lie on the angle bisectors of angles of  $\partial\omega$ ). This implies that the lengths of the legs of those trapezoids are bounded from below and above by  $c\delta$  and  $C\delta$ , for some real constants  $C, c > 0$ .

Let us make the set of vertices denser if necessary (by introducing more vertices along all sides, uniformly on each side, and dividing zones in more trapezoids) such that the

length of all new sides are less than  $\delta$ , but not less than  $\delta/2$ . This denser sets of vertices define large number of smaller trapezoids. One can easily see that the lengths of legs are not greater than the largest leg of original thin and long trapezoids, nor less than the smallest leg of original thin and long trapezoids. Thus all sides of small trapezoids are bounded from below and above by  $c\delta$  and  $C\delta$ , for some real constants  $C, c > 0$ . Note that even diagonals of small trapezoids are bounded in the same way (with adjusting the constants if necessary: they are bounded from below by height of length  $\delta$ , and above by sum of lengths of two sides of the trapezoid).

The idea of the algorithm is to build approximation functions  $(\psi_\delta, \mathbf{S}_\delta)$  with following properties:

- along the line  $\ell_0 = \partial\omega$  functions  $\psi_\delta$  take the value of  $\theta$ ;
- along the line  $\ell_1$  functions  $\psi_\delta$  are linear approximations of the function  $\theta$ ;
- along the line  $\ell_2$  functions  $\psi_\delta$  are linear approximations of the function  $\psi_\infty$ ;
- in  $\omega'$  we perform first 5 steps of the proof (as if we are taking in consideration only  $\omega'$ ), with piecewise affine approximation of  $\psi_\infty$  at  $\ell_2 = \partial\omega'$  - we know that such  $\psi_\delta|_{\omega'}$  is arbitrarily close approximation of  $\psi_\infty$ , and the same for  $\mathbf{S}_\delta$ ;
- in the zone  $[\ell_1, \ell_2]$  we first take the triangulation with given vertices from  $\ell_1$  and  $\ell_2$ . For  $\psi_\delta$  we take the linear approximation with respect to the linear approximations on the lines  $\ell_2$  (of  $\psi_\infty$ ) and  $\ell_1$  (of  $\theta$ ) and for  $\mathbf{S}_\delta$  we take any (piecewise on the triangulation) constant value for which the determinant condition is satisfied;
- in the zone  $[\ell_0, \ell_1]$  for  $\psi_\delta$  we are more careful (we will be more precise later) - the idea is to have  $\nabla'\psi_\delta \approx \nabla'\theta$ , and thus the choice  $\mathbf{S}_\delta = \mathbf{I}$  satisfies the determinant condition;
- thus  $\mathbf{S}_\delta$  is either continuous with the value  $\mathbf{I}$  at the boundary of particular area or just piecewise constant - in that case we perform the trick from the Step 5 to make  $\mathbf{S}_\delta$  continuous (in particular this is important in the zone  $[\ell_1, \ell_2]$ ).

We still need to define  $\psi_\delta$  in the buffer zone  $[\ell_0, \ell_1]$  and prove that when  $\delta \rightarrow 0$  the error  $\|\psi_\delta - \psi_\infty\|_{W^{1,4}([\ell_0, \ell_2]; \mathbb{R}^3)}^4$  goes to zero and that  $\det \begin{bmatrix} \mathbf{a}_3 & \partial_1 \psi_\delta & \partial_2 \psi_\delta \end{bmatrix} \geq 0$  in  $[\ell_0, \ell_1]$ .

Then the proof will be over since the part of the error  $\|\psi_\delta - \psi_\infty\|_{W^{1,4}(\omega'; \mathbb{R}^3)}^4$  is arbitrarily close to zero due to first 5 steps of the proof and the error  $\|\mathbf{S}_\delta - \mathbf{S}_\infty\|_{L^2(\omega; \text{SO}(3))}$  is close to zero due to first 5 steps of the proof (for  $\omega'$ ) and the small area of the buffer zone  $[\ell_0, \ell_2]$ . The construction makes the condition  $\det \begin{bmatrix} \mathbf{S}_\delta \mathbf{a}_3 & \partial_1 \psi_\delta & \partial_2 \psi_\delta \end{bmatrix} \geq 0$  satisfied and Dirichlet boundary conditions are trivially satisfied as well.

Let us now focus on  $\psi_\delta$  in  $[\ell_0, \ell_1]$ . Let us take a trapezoid  $A_0 B_0 B_1 A_1$  in  $[\ell_0, \ell_1]$  such that its vertices  $A_0, B_0 \in \ell_0$  and  $A_1, B_1 \in \ell_1$  are ending points of sides of polygonal lines  $\ell_0$  and  $\ell_1$  respectively. Let us firstly construct function  $\xi$  on that trapezoid (and analogously on the zone  $[\ell_0, \ell_1]$ ). Let  $\xi = 0$  on sides  $\overline{A_0 B_0}$ ,  $\overline{A_0 A_1}$  and  $\overline{B_1 B_0}$ . On the side  $\overline{A_1 B_1}$  let it be the difference between the function  $\theta$  and linear approximation of that function (with respect to points  $A_1$  and  $B_1$ ). One can easily see that the value of  $\xi$  and its derivative along  $\overline{A_1 B_1}$  can be bounded by  $\|\nabla' \theta\|_{L^\infty(\omega; \mathbb{R}^{3 \times 2})} |A_1 B_1|$  and  $\|D^2 \theta\|_{L^\infty(\omega; \mathbb{R}^{3 \times 2 \times 2})} |A_1 B_1|$ , respectively. Since  $\theta \in C^2(\omega; \mathbb{R}^3)$  and all sides in  $\ell_1$  are of length which is less than delta, those values are bounded by  $C\delta$ , with  $C > 0$  some real constant. One can simply define  $\xi$  with the same bound on its value and derivatives inside the trapezoid  $A_0 B_0 B_1 A_1$  as well (by linear functions on properly chosen lines connecting points on  $\ell_0$  (value there is 0) and  $\ell_1$  (value there is prescribed difference of  $\theta$  and its linear approximation using points  $A_1$  and  $B_1$ )). Now we define  $\psi_\delta$  on that trapezoid (and the whole zone  $[\ell_0, \ell_1]$  as well) as  $\psi_\delta = \theta - \xi$ . Since  $\|\nabla' \xi\|_{L^\infty(\omega; \mathbb{R}^{3 \times 2})} \leq C\delta$ , we have that in each point  $\mathbf{y}' \in [\ell_0, \ell_1]$  it holds  $\|\partial_\alpha \psi_\delta - \mathbf{a}_\alpha\|_{L^\infty(\omega; \mathbb{R}^3)} \leq C\delta$ ,  $\alpha = 1, 2$ . Since  $\det \begin{bmatrix} \mathbf{a}_3 & \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \sqrt{a} > 0$  in all points  $\mathbf{y}' \in \omega$  (moreover, the determinant is bounded from below by a positive constant because of (5.2.1)), due to continuity of the determinant for  $\delta$  small enough we have  $\det \begin{bmatrix} \mathbf{a}_3 & \partial_1 \psi_\delta & \partial_2 \psi_\delta \end{bmatrix} > 0$  on  $[\ell_0, \ell_1]$ . Thus on this zone the choice  $\mathbf{S}_\delta = \mathbf{I}$  fulfils the determinant condition.

Finally, let us prove that the error  $\|\psi_\delta - \psi_\infty\|_{W^{1,4}([\ell_0, \ell_2]; \mathbb{R}^3)}^4$  goes to zero as  $\delta \rightarrow 0$ . For this it is sufficient to prove claim only for the zone  $[\ell_1, \ell_2]$ , since in the zone  $[\ell_0, \ell_1]$  it is controlled by the  $\infty$ -norm of the function  $\xi$  and the measure of the set  $[\ell_0, \ell_1]$ . In the zone  $[\ell_1, \ell_2]$  we approach similarly, we will prove that the values  $\|\psi_\delta\|_{L^\infty([\ell_1, \ell_2]; \mathbb{R}^3)}$  and  $\|\nabla' \psi_\delta\|_{L^\infty([\ell_1, \ell_2]; \mathbb{R}^{3 \times 2})}$  are bounded independently of  $\delta$  - then the claim is a consequence of the measure of the set  $[\ell_1, \ell_2]$  being of order  $\delta$ .

For the values  $\psi_\delta$  the estimate is trivial, since in that zone all values are convex com-



binations of the values of functions  $\psi_\infty$  and  $\theta$  which are uniformly bounded.

For derivatives, we have to bound the value  $\|\nabla'\psi_\delta(\mathbf{y}')\|_F$  on each triangle, uniformly on all triangles and independently of  $\delta$ . Since  $\psi_\delta$  is piecewise affine function,  $\nabla'\psi_\delta(\mathbf{y}')$  is piecewise constant. We firstly note that since all triangles in  $[\ell_1, \ell_2]$  have sides bounded from above and below by  $C\delta$  and  $c\delta$  (since the sides and the diagonals of trapezoids have the same property), the angles in those triangles are uniformly (independently of  $\delta$ ) bounded from above and below by constants distinct from 0 and  $\pi$ . This allows us to bound  $\|\nabla'\psi_\delta(\mathbf{y}')\|_F$  by derivatives of  $\psi_\delta$  along any two sides of the triangle on which they are defined.

All derivatives along sides can be found by quotients of the form  $\frac{\psi_\delta(A') - \psi_\delta(B')}{|A'B'|}$ , where  $A', B'$  belong to  $\ell_1 \cup \ell_2$ . Let us choose a fixed point  $P$  on  $\partial\omega$  (depending on  $A'$  and  $B'$ ). It is possible to do it in such way that the uniform bounds  $|A'P|, |B'P| \leq C'\delta$  still hold (for some constant  $C' > 0$ ). The denominator  $|A'B'|$  in the derivative is bounded from below by  $c\delta$ . The value of  $\psi_\delta(A')$  has either value  $\psi_\infty(A')$  or value  $\theta(A')$ . Since both functions  $\psi_\infty$  and  $\theta$  have uniformly bounded first derivatives and since they both are equal to  $\theta$  on  $\partial\omega$  by the Dirichlet boundary condition, we have  $\|\psi_\delta(A') - \theta(P)\| \leq C''|A'P| \leq C'''\delta$ , i.e.  $\psi_\delta(A') = \theta(P) + \mathcal{O}(\delta)$ . Since all can be done for  $B'$  as well, we have that

$$\left\| \frac{\psi_\delta(A') - \psi_\delta(B')}{|A'B'|} \right\| = \frac{\|\theta(P) - \theta(P) + \mathcal{O}(\delta)\|}{|A'B'|} \leq \frac{\mathcal{O}(\delta)}{c\delta} \leq \mathcal{O}(1),$$

which proves that the derivatives of  $\psi_\delta$  are bounded in  $[\ell_1, \ell_2]$ .

This finally completes the proof. □

**Remark A.2.** As already noted in the beginning of the Step 6 of the proof the statement of Lemma A.1 holds also in the case  $\gamma_0 \neq \partial\omega$ , but for  $\theta$  which is piecewise affine at the boundary at  $\gamma_0$ , including the trivial case  $\gamma_0 = \emptyset$ .

# BIBLIOGRAPHY

- [1] E. Acerbi, G. Buttazzo and D. Percivale, *Thin inclusions in linear elasticity: a variational approach*, J Reine Angew Math **386** (1988), 99–115. [↑ 7](#).
- [2] I. Aganović, J. Tambača and Z. Tutek, *A note on reduction of dimension for linear elliptic equations*, Glasnik Matematicki **41** (2006), 77–88. [↑ 21](#).
- [3] S. Anicic, *Polyconvexity and existence theorem for nonlinearly elastic shells*. J. Elasticity **132** (2018), 161–173. [↑ 7](#).
- [4] S. Anicic, *Existence theorem for a first-order Koiter nonlinear shell model*. Discrete Contin. Dyn. Syst. Ser. S **12** (2019), 1535–1545. [↑ 7](#).
- [5] S. Anicic, H. Le Dret, A. Raoult, *The infinitesimal rigid displacement lemma in Lipschitz co-ordinates and application to shells with minimal regularity*. Math. Methods Appl. Sci. **27** (2004), no. 11, 1283–1299. [↑ 86, 87](#).
- [6] S.S. Antman, *Nonlinear problems of elasticity*. Applied Mathematical Sciences, 107. Springer-Verlag, New York, 1995. [↑ 6, 87, 103](#).
- [7] B. Belgacem, *Une méthode de  $\Gamma$ -convergence pour modèle de membrane non linéaire*. Academie des Sciences Paris Comptes Rendus Serie Sciences Mathematiques. (1997). 324. [↑ 59, 66](#).
- [8] A. L. Bessoud, F. Krasucki and G. Michaille, *Multi-materials with strong interface: variational modelings*, Asymptot. Anal. **61** (2009), 1–19. [↑ 7, 73, 76, 77](#).
- [9] A.-L. Bessoud, F. Krasucki and M. Serpilli, *Asymptotic analysis of shell-like inclusions with high rigidity*, J. Elasticity **103** (2011), 153–172. [↑ 7](#).

- [10] A. Bessoud, F. Krasucki and M. Serpilli, *Plate-like and shell-like inclusions with high rigidity*, Comptes Rendus de l'Academie des Sciences. Serie 1, Mathematique. 346. (2008) [↑ 7](#).
- [11] M. Bîrsan, I.D. Ghiba, R. Martin and P. Neff, *Refined dimensional reduction for isotropic elastic Cosserat shells with initial curvature*, Math. Mech. Solids **24** (2019), 4000–4019. [↑ 7](#).
- [12] M. Bîrsan and P. Neff, *Existence of minimizers in the geometrically non-linear 6-parameter resultant shell theory with drilling rotations*, Math. Mech. Solids **19** (2014), 376–397. [↑ 87](#).
- [13] D. Blanchard and G. Griso, *A simplified model for elastic thin shells*, Asymptot. Anal. **76** (2012), 1–33. [↑ 6](#).
- [14] A. Blouza and H. Le Dret, *An up-to-the-boundary version of Friedrichs's lemma and applications to the linear Koiter shell model*. SIAM J. on Math. Anal. **33** (2001), 877–895. [↑ 86](#).
- [15] A. Blouza and H. Le Dret, *Existence and uniqueness for the linear Koiter model for shells with little regularity*. Quart. Appl. Math. **57** (1999), 317–337. [↑ 86](#).
- [16] A. Blouza and H. Le Dret, *Naghdi's shell model: existence, uniqueness and continuous dependence on the midsurface*. J. Elasticity **64** (2001), no. 2-3, 199–216. [↑ 86](#), [87](#).
- [17] S.C. Brenner and L.R. Scott. *The mathematical theory of finite element methods*, Volume 15. Springer, 1994. [↑ 140](#), [168](#).
- [18] Caillerie D., *The effect of a thin inclusion of high rigidity in an elastic body*, Math. Methods Appl. Sci., 2(3), 1980., 251–270. [↑ 7](#).
- [19] P.G. Ciarlet, *Mathematical elasticity. Vol. I. Three-dimensional elasticity*, North-Holland Publishing Co., Amsterdam, 1988. [↑ 25](#).

- [20] P.G. Ciarlet, *Mathematical elasticity. Vol. II. Theory of plates*. Studies in Mathematics and its Applications, 27. North-Holland Publishing Co., Amsterdam, 1997.  
↑ 6, 7, 21.
- [21] P.G. Ciarlet, *Mathematical elasticity. Vol. III. Theory of shells*. Studies in Mathematics and its Applications, 29. North-Holland Publishing Co., Amsterdam, 2000.  
↑ 6, 15, 25, 36, 87, 98, 99.
- [22] P.G. Ciarlet, *Un modèle bi-dimensionnel non linéaire de coques analogue à celui de W.T. Koiter*, C. R. Acad. Sci. Paris Ser. I **331** (2000), 405–410. ↑ 6.
- [23] P.G. Ciarlet, H. Le Dret and R. Nzingwa, *Junctions between three-dimensional and two-dimensional linearly elastic structures*, J. Math. Pures Appl. **68** (1989), 261–295. ↑ 7.
- [24] P.G. Ciarlet, V. Lods and B. Miara, Asymptotic analysis of linearly elastic shells. II. Justification of flexural shell equations. Arch. Rational Mech. Anal. **136** (1996), no. 2, 163–190. ↑ 17.
- [25] P.G. Ciarlet and C. Mardare, *An existence theorem for a two-dimensional nonlinear shell model of Koiter's type* Math. Models Methods Appl. Sci. **28** (2018), 2833–2861. ↑ 7, 87.
- [26] P.G. Ciarlet and C. Mardare, *A nonlinear shell model of Koiter's type*, C. R. Math. Acad. Sci. Paris **356** (2018), 227–234. ↑ 6, 87.
- [27] P.G. Ciarlet and C. Mardare, *An intrinsic formulation of the Kirchhoff-Love theory of linearly elastic plates*, Anal. Appl. (Singap.) **16** (2018), 565–584. ↑ 6.
- [28] P.G. Ciarlet and C. Mardare, *An introduction to shell theory*, in: Differential Geometry: Theory and Applications, Ser. Contemp. Appl. Math. CAM, Vol. 9, Higher Education Press, Beijing, 2008, pp. 94–184. ↑ 6.
- [29] P.G. Ciarlet and C. Mardare, *Asymptotic justification of the intrinsic equations of Koiter's model of a linearly elastic shell*, C. R. Acad. Sci. Paris **357** (2019), 99–110.  
↑ 6.

- [30] S. Conti and F. Maggi, *Confining thin elastic sheets and folding paper*, Arch. Ration. Mech. Anal. **187** (2008). 1–48. [↑](#) [6](#), [59](#), [66](#), [88](#), [132](#), [133](#), [134](#), [165](#).
- [31] S. Conti, F. Maggi and S. Müller, *Rigorous derivation of Föppl’s theory for clamped elastic membranes leads to relaxation*, SIAM J. Math. Analysis **38** (2006), 657–680 [↑](#) [6](#).
- [32] B. Dacorogna, *Direct methods in the calculus of variations. Second edition*. Springer, New York, 2008. [↑](#) [124](#).
- [33] S. Dumont, F. Lebon and R. Rizzoni, *Imperfect interfaces with graded materials and unilateral conditions: theoretical and numerical study*, Math. Mech. Solids **23** (2018), 445–460. [↑](#) [7](#).
- [34] I. Ekeland and R. Témam, *Convex analysis and variational problems*. Translated from the French. Corrected reprint of the 1976 English edition. SIAM, Philadelphia, 1999. [↑](#) [115](#).
- [35] E. El Bachari, *Modelling of a 3d-2d junction in nonlinear elasticity by  $G$ -convergence* C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), 1241–1246. [↑](#) [7](#).
- [36] El Jarroudi, Mustapha, *Asymptotic analysis of contact problems between an elastic material and thin-rigid plates*, Applicable Analysis. **89** (2010). 693–715. [↑](#) [7](#).
- [37] D. Fox, A. Raoult and J.C. Simo, *A justification of nonlinear properly invariant plate theories*, Arch. Ration. Mech. Anal. **124** (1993), 157–199. [↑](#) [6](#).
- [38] G. Friesecke, R.D. James and S. Müller, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from the three-dimensional elasticity*, Comm. Pure Appl. Math. **55** (2002), 1461–1506. [↑](#) [6](#), [59](#), [66](#), [162](#).
- [39] G. Friesecke, R.D. James and S. Müller, *A hierarchy of plate models derived from nonlinear elasticity by  $\Gamma$ -convergence*, Arch. Rat. Mech. Anal. **180** (2006), 183–236. [↑](#) [6](#), [53](#), [166](#).

- [40] G. Friesecke, R.D. James, M.G. Mora and S. Müller, *Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by Gamma-convergence*, C. R. Math. Acad. Sci. Paris **336** (2003), 697–702. ↑ 6, 87, 88, 92, 97, 98, 123, 128, 130, 165.
- [41] J.F. Ganghoffer, A. Brillard and R. De Borst, *Description of the mechanical behaviour of micropolar adhesives*, Math. Comput. Modell. **27** (1998), 23–49. ↑ 7.
- [42] G. Geymonat and F. Krasucki, *Asymptotic analysis of the behaviour of two bonded plates*, C. R. Acad. Sci. IIB **325** (1997), 307–314. ↑ 7.
- [43] G. Geymonat, F. Krasucki and S. Lenci, *Mathematical analysis of a bonded joint with a soft thin adhesive*, Math Mech Solids **4** (1999), 201–225. ↑ 7.
- [44] T.J.R. Hughes and F. Brezzi, *On drilling degrees of freedom*. Computer Methods in Applied Mechanics and Engineering **72** (1989), 105–121. ↑ 87.
- [45] T.J.R. Hughes, A. Masudg and I. Harari, *Numerical assessment of some membrane elements with drilling degrees of freedom*. Computers & Structures **55** (1995), 297–314. ↑ 87.
- [46] A.M. Khludnev, L. Faella and T.S. Popova, *Junction problem for rigid and Timoshenko elastic inclusions in elastic bodies*, Math. Mech. Solids **22** (2017), no. 4, 737–750. ↑ 7.
- [47] W.T. Koiter, *On the nonlinear theory of thin elastic shells. I, II, III*, Nederl. Akad. Wetensch. Proc. Ser. B **69** (1966), 1–17, 18–32, 33–54. ↑ 6, 87, 99.
- [48] M. Kosor and J. Tambača, *Nonlinear bending-torsion model for curved rods with little regularity*, Math. Mech. Solids **22** (2017), 708–717. ↑ 87.
- [49] F. Krasucki, A. Münch and Y. Ousset, *Mathematical analysis of nonlinear bonded joint models*, Math Models Meth Appl Sci **14** (2004), 1–22. ↑ 7.
- [50] H. Le Dret, *Well-posedness for Koiter and Naghdi shells with a  $G_1$ -midsurface*. Anal. Appl. (Singap.) **2** (2004), no. 4, 365–388. ↑ 87.

- [51] H. Le Dret and A. Raoult, *The membrane shell model in nonlinear elasticity: a variational asymptotic derivation*, J. Nonlinear Sci. **6** (1996), 59–84. ↑ 6, 88, 92, 112, 118, 122, 123, 165.
- [52] H. Le Dret and A. Raoult, *The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity*, J. Math. Pures Appl. **74** (1995), 549–578. ↑ 6, 59, 60, 66, 112, 118, 163.
- [53] H. Le Dret and A. Raoult, *The quasiconvex envelope of the Saint Venant–Kirchhoff stored energy function*, Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), 1179–1192. ↑ 6.
- [54] H. Le Dret and A. Raoult, *Variational convergence for nonlinear shell models with directors and related semicontinuity and relaxation results*, Arch. Rational Mech. Anal. **154** (2000), 101–134. ↑ 6.
- [55] F. Lebon and R. Rizzoni, *Asymptotic analysis of a thin interface: the case involving similar rigidity*, Int J Eng Sci **48** (2010), 473–486. ↑ 7.
- [56] F. Lebon and R. Rizzoni, *Asymptotic behavior of a hard thin linear interphase: an energy approach*, Int J Solids Struct **48** (2011), 441–449. ↑ 7.
- [57] M. Lewicka, *A note on convergence of low energy critical points of nonlinear elasticity functionals, for thin shells of arbitrary geometry*, ESAIM Control Optim. Calc. Var. **17** (2009), 493–505. ↑ 6.
- [58] M. Lewicka, M.G. Mora and R. Pakzad, *Shell theories arising as low energy gamma-limit of 3d nonlinear elasticity*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) **9** (2010), 1–43. ↑ 6.
- [59] M. Lewicka, M.G. Mora and R. Pakzad, *The matching property of infinitesimal isometries on elliptic surfaces and elasticity of thin shells*, Arch. Ration. Mech. Anal. **200** (2011), 1023–1050. ↑ 6.
- [60] M. Lewicka and R. Pakzad, *The infinite hierarchy of elastic shell models: some recent results and a conjecture*, Fields Inst. Commun. **64**, 2009, 407–420. ↑ 6, 53, 166.

- [61] M. Ljulj and J. Tambača, *A Naghdi type nonlinear model for shells with little regularity*, submitted, 2020. [↑](#) 6, 86, 109.
- [62] M. Ljulj and J. Tambača, *3D structure – 2D plate interaction problem*, *Mathematics and Mechanics of Solids* **24** (2019), 3354–3377. [↑](#) 6.
- [63] M. Ljulj and J. Tambača, *Numerical investigation of the 2d–1d structure interaction model*, submitted, 2019. [↑](#) 1, 4.
- [64] I. Malakhova-Ziablova, G. Panasenko and R. Stavre, *Asymptotic analysis of a thin rigid stratified elastic plate–viscous fluid interaction problem*, *Appl. Anal.* **95** (2016), no. 7, 1467–1506. [↑](#) 7.
- [65] C. Mardare, *Nonlinear shell models of Kirchhoff-Love Type: existence theorem and comparison with Koiter’s model*, *Acta Mathematicae Applicatae Sinica, English Series*. 35 (2019), 3–27. [↑](#) 6.
- [66] C. Mardare, *On the derivation of nonlinear shell models from three-dimensional elasticity*, *Rev. Roumaine Maths. Pures Appl.* **53** (2008), 499–522. [↑](#) 6.
- [67] P.M. Naghdi, *The theory of shells and plates*, *Handbuch der Physik*, vol. VIa/2, Springer-Verlag, New York, 1972, 425–640. [↑](#) 6, 12.
- [68] S.A. Nazarov, *Korn’s inequalities for junctions of elastic bodies with thin plates*. (Russian) *Sibirsk. Mat. Zh.* **46** (2005), 876–889; translation in *Siberian Math. J.* **46** (2005), 695–706. [↑](#) 27.
- [69] P. Neff, *A geometrically exact Cosserat-shell model including size effects, avoiding degeneracy in the thin shell limit. Part I: Formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus*, *Cont. Mech. Thermodyn.* **16** (2004), 577–628. [↑](#) 7.
- [70] P. Neff, *Geometrically exact Cosserat theory for bulk behaviour and thin structures. Modelling and mathematical analysis*. Signatur HS 7/0973. Habilitationsschrift, Universitäts- und Landesbibliothek, Technische Universität Darmstadt, Darmstadt, 2004. [↑](#) 7.



- [71] P. Neff, *A geometrically exact planar Cosserat shell-model with microstructure: Existence of minimizers for zero Cosserat couple modulus*, Math. Mod. Meth. Appl. Sci. **17** (2007), 363–392. [↑ 7](#).
- [72] P. Neff and K. Chelmiński, *A geometrically exact Cosserat shell-model for defective elastic crystals. Justification via Gamma-convergence*, Interfaces Free Bound. **9** (2007), 455–492. [↑ 7](#).
- [73] G.P. Panasenko and R. Stavre, *Asymptotic analysis of a viscous fluid–thin plate interaction: periodic flow*, Math. Models Methods Appl. Sci. **24** (2014), 1781–1822. [↑ 7](#).
- [74] O. Pantz, *On the justification of the nonlinear inextensional plate model*, Arch. Ration. Mech. Anal. **167** (2003), 179–209. [↑ 6](#).
- [75] M. L. Raffa, F. Lebon and R. Rizzoni, *Derivation of a model of imperfect interface with finite strains and damage by asymptotic techniques: an application to masonry structures*, Meccanica **53** (2018), 1645–1660. [↑ 7](#).
- [76] O. Rodrigues, *Des lois géométriques qui régissent les déplacements d’ un système solide dans l’ espace, et de la variation des coordonnées provenant de ces déplacements considérées indépendamment des causes qui peuvent les produire*, J. Math. Pures Appl. **5** (1840), 380–440. [↑ 168](#).
- [77] K. Schmidt and S. Tordeux, *Asymptotic modelling of conductive thin sheets*, Z. Angew. Math. Phys. **61** (2010), no. 4, 603–626. [↑ 7](#).
- [78] A. Selvadurai, *A geophysical application of an elastostatic contact problem*, Mathematics and Mechanics of Solids, **18**(2) (2013), 192–203. [↑ 7](#).
- [79] M. Serpilli, *Asymptotic analysis of a multimaterial with a thin piezoelectric interphase*, Meccanica **49** (2014) 1641–1652. [↑ 7](#).
- [80] M. Serpilli, *Asymptotic interface models in magneto-electro-thermo-elastic composites*, Meccanica **52** (2017), 1407–1424. [↑ 7](#).

- [81] M. Serpilli, *Mathematical modeling of weak and strong piezoelectric interfaces*, J. Elasticity **121** (2015), 235–254. [↑ 7](#).
- [82] M. Serpilli, *On modeling interfaces in linear micropolar composites*, Math. Mech. Solids **23** (2018), 667–685. [↑ 7](#).
- [83] M. Serpilli and S. Lenci, *An overview of different asymptotic models for anisotropic three-layer plates with soft adhesive*, Int. J. Solid. Struct. **81** (2015). [↑ 7](#).
- [84] D. Steigmann, *Koiter’s shell theory from the perspective of three-dimensional non-linear elasticity*, J. Elasticity **111** (2013), 91–107. [↑ 6](#).
- [85] J. Tambača, *A note on the ”flexural” shell model for shells with little regularity*, Adv. Math. Sci. Appl. **16** (2006), 45–55. [↑ 15, 86](#).
- [86] J. Tambača, *A new linear shell model for shells with little regularity*, J. Elasticity **117** (2014), 2, 163–188. [↑ 15, 17, 86, 99](#).
- [87] J. Tambača and Z. Tutek, *A new linear Naghdi type shell model for shells with little regularity*, Appl. Math. Model. **40** (2016), 10549–10562. [↑ 5, 8, 12, 15, 16, 86, 87, 88, 99, 100, 101, 102, 103, 165](#).

# CURRICULUM VITAE

Matko Ljulj was born on 29<sup>th</sup> of October 1992 in Zagreb, Croatia, where he finished primary and secondary school. During secondary school he attended mathematical competitions, winning 1<sup>st</sup> prize on Croatian Mathematical Competition three times, and one Silver and two Bronze Medals at the International Mathematical Olympiad.

In 2011 he started studies at Department of Mathematics, Faculty of Science, University of Zagreb, where he obtained his master's degree in 2016 with master thesis *Numerical approximation of poroelastic shell model* under supervision of Prof. Dr. Josip Tambača. During his studies he attended undergraduate mathematical competitions, including International Mathematical Competition (two First Prizes and one Second Prize) and Vojtěch Jarník International Mathematical Competition (3<sup>rd</sup> place in 2013).

In 2016 he started working as a teaching assistant and enrolled his PhD studies at the same faculty under supervision of Prof. Dr. Josip Tambača. He attended five conferences and workshops, where he gave three talks and presented one poster. He had six short scientific visits to mathematical departments at TU Berlin and TU Darmstadt, Germany, collaborating with Prof. Dr. V. Mehrmann, PD Dr. K. Schmidt and Dr. A. Semin. During those visits he gave three talks. He is an active member of the *Seminar for differential equations and numerical analysis* at Department of Mathematics, Faculty of Science, University of Zagreb.

He co-authored nine papers, four of which are published:

- (1) S. Čanić, M. Galović, M. Ljulj, J. Tambača, A dimension-reduction based coupled model of mesh-reinforced shells, *SIAM Journal of Applied Mathematics* 77 (2017) 2, 744-769.
- (2) M. Ljulj, J. Tambača, Iterative methods for solving a poroelastic shell model of

Naghdi's type, M2AS 40 (2017), 12; 4425-4435.

- (3) S. Čanić, M. Galić, M. Ljulj, J. Tambača, Y. Wang, Analysis of a linear 3D fluid–mesh–shell interaction problem, Zeitschrift fur angewandte mathematik und physik 70 (2019), 2; 44, 38pp.
- (4) M. Ljulj, J. Tambača, 3D structure – 2D plate interaction problem, Mathematics and Mechanics of Solids 24 (2019), 3354-3377.